Extending characterizations of truthful mechanisms from subdomains to domains.

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Abstract. The already extended literature in combinatorial auctions, public projects and scheduling demands a more systematic classification of the domains and a clear comparison of the results known. Connecting characterization results for different settings and providing a characterization proof using another characterization result as a black box without having to repeat a tediously similar proof is not only elegant and desirable, but also greatly enhances our intuition and provides a classification of different results and a unified and deeper understanding. Characterizing the mechanisms for the domains of combinatorial auctions and scheduling unrelated machines are two outstanding problems in mechanism design. Since the scheduling domain is essentially the subdomain of combinatorial auctions with additive valuations, we consider whether one can extend a characterization of a subdomain to a domain. This is possible for two players (and for n-player mechanisms that satisfy stabilty) if the only truthful mechanisms for the sub-domain are the affine maximizers. Although this is not true for scheduling because besides the affine maximizers there are other truthful mechanisms (the threshold mechanisms), we still show that the truthful mechanisms that allocate all goods of practically any domain which is strictly superdomain of additive combinatorial auctions are only the affine maximizers.

1 Introduction

Our results and motivation. Roberts [12](1979) gives an elegant proof, which shows that the only truthful mechanisms for the Unrestricted domain are the affine maximizers. He also gets the Gibbard-Sattherwhaite Theorem (1973) for voting systems as a corollary. For more "restricted" multi-parameter domains, there exist truthful mechanisms other than affine maximizers (see e.g. [16, 11, 4]). An important question, posed in [19, 15], is to determine how much we need to restrict the domain in order to admit truthful mechanisms different than the affine maximizers. Here we show that for the case of two players, the transition domain is the additive combinatorial auctions(/scheduling) domain: We show that if we slightly enrich the possible valuations, the threshold mechanisms involved in the characterization [4] seize to be truthful and the only truthful mechanisms that remain are the affine maximizers.

In this work we address but only partially answer the following very important strengthenings of this question: In which way should we restrict the domain? Which domains have the same characterization? Can we classify the domains in a hierarchy in terms of how difficult it is to characterize them (if their characterization is the same) and how rich are the mechanisms allowed (else)? Every time we achieve to characterize a more difficult domain do we automatically get a proof for domains that are lower in this hierarchy? For which domains can we establish a bijection between the mechanisms involved in their characterization? This paper gives some explanations we would have liked to find, back when we started working on characterization results and wondered what do the results about other slightly different domains tell us about the domain we were primarily interested in.

A crucial observation is: the more "unrestricted" the domain of valuations, the fewer the possible truthful mechanisms. An intuitive explanation for this is that in larger domains there are more inputs that need to satisfy the conditions for truthfulness. On the other hand, *this intuition may be misleading*: Given that a sub-domain admits as truthful mechanisms only the affine maximizers does not immediately imply that the domain also admits the same mechanisms; there may be other mechanisms which when restricted to sub-domain are exactly the affine maximizers. In particular, we don't know whether this is possible for more that 2 players. However, for the case of 2 payers we verify this intuition: A complete characterization for the scheduling problem, where the valuations are heavily restricted to additive ones, involves a combination of affine minimizers and threshold mechanisms. On the other hand the characterization for all it's super-domains can be easily derived from it; this derivation is much easier and clearer and involves only affine maximizers.

We provide a single characterization proof for any super-domain of this slight enrichment of the additive domain. One of these super-domains is the domain of 2-player combinatorial auctions with sub-modular valuations (that allocate all items), an important domain about which no characterization was previously known, but also the already known characterizations for 2-player combinatorial auctions [16] and combinatorial auctions with sub-additive valuations [11]. Our approach also goes through for n-player stable mechanisms.

Our work proposes a common general framework that classifies a multitude of different domains. If you prove a characterization of truthful mechanisms for a specific (2-player that allocates all items, or n-player stable) domain in terms of affine maximizers and threshold mechanisms, you can plug your Theorem as black box in our theorems here and get a characterization of all its super-domains. So it is more important to characterize a domain with a rich class of super-domains. The notions of translated domains and bijections between domains and the tools we develop might be further useful.

Related work. The starting point of characterization attempts goes back to Robert's [12] result. Many papers tried to extend this very elegant proof [17, 11, 20], while others tried different proof techniques [16, 4, 9, 17]. (As the literature in combinatorial auctions is vast we refer the reader to [19] Chapter 11 and the references within and mention here only some recent results.) An important direction is the quest for polynomial-time algorithms. Computational complexity impossibility results for maximal in range mechanisms where shown in [2, 8]. Dobzinski [7] shows that every universally truthful randomized mechanism for combinatorial auctions with submodular valuations that provides an approximation ratio of $m^{\frac{1}{2}-\epsilon}$ must use exponentially many value queries. Krysta and Ventre show that if verification is introduced sub-modular combinatorial auctions become tractable [14]. Many more interesting results arise when one considers randomized mechanisms. Another very well-studied relevant problem is that of multi-unit auctions, and one of our proofs here goes along the same lines as a proof from [6].

Nisan and Ronen introduced the mechanism-design version of the scheduling problem on unrelated machines [18, 5, 13]. For the case of two machines [11] Dobzinski and Sundararajan characterized all mechanisms with finite approximation ratio for the objective of minimizing the makespan, while [4] gave a characterization regardless of approximation ratio of decisive truthful mechanisms (which also implies a characterization of additive combinatorial auctions that we will use here) in terms of affine minimizers and threshold mechanisms.

1.1 Definitions and preliminaries

Stability is without loss of generality for 2-player auctions that allocate all items, unrestricted domains and combinatorial public projects. A mechanism is called *stable* if the following holds: *For fixed valuations v_{-i}, the allocation a_i of player i determines uniquely the* *allocation a_{-i} of the other players.* (In other words: Fix v_{-i}, then for all v_i for which player i has allocation a_i the allocation a_{-i} is the same.) *Stability can be assumed without loss of generality for unrestricted domains, combinatorial public projects and 2-player auctions where all items are allocated.* It is too restrictive for combinatorial auctions with $n \geq 3$ players (see [16] Example 4), however all known characterization results [12, 16, 20, 11, 4, 17, 11] heavily rely on stability, or characterize domains where stability can be assumed essentially without loss of generality. Stability is implied by S-MON or IIA (see [16, 1, 11] for a discussion on these conditions and proofs).

Lemma 1. *For a truthful mechanism when v_{-i} is fixed: (a)The price $p_i(v_i, v_{-i})$ cannot depend directly on the declaration v_i of player i, but only on his allocation $a_i(v_i, v_{-i})$ and the declarations of the other players, that is $p_i(v_i, v_{-i}) = p_i(a_i(v_i, v_{-i}), v_{-i})$.*

(b) For every player i the outcome a_i satisfies $a_i(v_i, v_{-i}) \in \operatorname{argmax}_{a_i}\{v_i(a_i) - p_i(a_i, v_{-i})\}$ where the quantification is over all the alternatives that i can enforce for different v_i and fixed v_{-i}.

(c) If for fixed v_{-i} the regions where player i has assignment a_i and a'_i, share a common boundary, then any valuation v_i on tis boundary satisfies $v_i(a_i) - v_i(a'_i) = p_i(a_i, v_{-i}) - p_i(a'_i, v_{-i})$.

A matrix representation of finite domains. [3] We will denote any finite domain of valuations D as a set of matrices. We have one matrix for each valuation function $v = (v_1, \ldots, v_n) : A \to \mathbb{R}$ that belongs to the domain. This matrix has one column for each alternative $a \in A$ and one row for each player. Thus the valuation v_i of player i is a vector (row of the previous matrix) of numbers that has one coordinate for each possible alternative and we denote the set of all possible such vectors for player i by V_i. (The domain is the set of all possible inputs of the mechanism.)

Under this notation the domain of unrestricted valuations (for which a complete characterization is given in [12]) contains all possible matrices with real values.

We will say that S_i is a subdomain of V_i if the set of all possible valuation vectors S_i is a subset of V_i. We will say that $D = S_1 \times \ldots \times S_n$ is a subdomain of $D' = V_1 \times \ldots \times V_n$ if $D \subseteq D'$.
Affine transformations of domains. If D is the matrix representation of a domain we denote by $\lambda D + c$ the following affine transformation of D: Multiply the valuations of each player i by a positive constant λ_i and add a matrix of constants c, with one row c^i for each player and one column for each possible allocation. For example the following is an affine transformation of 2-player combinatorial auctions:

$$\begin{pmatrix} c^1_{\emptyset} & \lambda_1 v_1(\{1\}) + c^1_{\{1\}} & \lambda_1 v_1(\{2\}) + c^1_{\{2\}} & \lambda_1 v_1(\{1,2\}) + c^1_{\{1,2\}} \\ \lambda_2 v_2(\{1,2\}) + c^2_{\{1,2\}} & \lambda_2 v_2(\{2\}) + c^2_{\{2\}} & \lambda_2 v_2(\{1\}) + c^2_{\{1\}} & c^2_{\emptyset} \end{pmatrix}.$$

2 Our results

Derivation of the characterization of a domain from the characterization of one of its sub-domains. Suppose we know which mechanisms are truthful for a given domain, does this tell us which mechanisms are truthful for any super-domain of it? The first reaction may be: we can read the proofs and produce (tediously) similar ones. But then the mechanism for the bigger domain has to satisfy truthfulness for a superset of the input space. Are then perhaps the mechanisms for the bigger domain a subset of the mechanisms for the sub-domain? We have to be careful: it is true that if a mechanism is truthful for the bigger domain, then its restriction to the smaller domain is a truthful mechanism for the smaller domain (for which we assumed that we know a characterization). However it then remains to describe the mechanism for the additional inputs we allowed by enlarging the domain.

Theorem 1. *Let V be a sub-domain of the domain of unrestricted valuations and superdomain of the domain of additive valuations. If the only possible n-player stable mechanisms for V are affine maximizers, then the same holds for every super-domain of V.*[1]

We want to show that there is no other way to extend a mechanism, which is an affine maximizer for the smaller domain V, to the bigger domain other than an affine maximizer for the bigger domain. If we did not require the mechanism to be truthful, then there would be many possibilities to extend the mechanism to a mechanism that would not be an affine maximizer for the whole domain.

Note that in Theorem 1 we did not assume decisiveness, this is because Lemma 2 shows that by truthfulness the range of the mechanism for the bigger domain is the same as the range of it's restriction to the subdomain.

Lemma 2. *Let S_i be the domain of additive valuations, or any super-domain of it, and $S_i \subseteq V_i$. Consider a social choice function $f(\cdot, v_{-i}) : V_1 \times \ldots \times V_n \to A$ for fixed v_{-i}, and constrain it to the domain $S_1 \times \ldots \times S_n$. If the range of the restricted function is a set of alternatives A, then the same set of alternatives is also the range of the social choice function $f(\cdot, v_{-i})$ when it is constrained to the bigger domain $S_1 \times \ldots \times V_i \times \ldots \times S_n$.*

Lemma 3. *Start with an affine maximizer M defined for the domain of valuations $S_1 \times \ldots \times S_n$ and then consider the bigger domain $S_1 \times \ldots \times V_i \times \ldots \times S_n$ where V_i is such that $S_i \subseteq V_i$.*

If we concentrate on stable mechanisms, there is a unique way to extend M to a truthful mechanism for the bigger domain, namely an affine maximizer defined by the same λ, γ as M.

Affine transformations of domains. Note that the next theorem holds for any choice of the domain D, and not only for the domain of additive valuations. This theorem implies that if we characterize all possible mechanisms for a domain of valuations D then the same characterization holds for all domains we get by translating D.

Theorem 2. *There is a bijection between the mechanisms for D and the mechanisms of $\lambda D + c$. That is the mechanism with the same allocation and payments $p' = \lambda \cdot p + c$ is also truthful for $\lambda D + c$. This holds for any number of players n.*

Threshold mechanisms and their payments. The characterization in [4] reveals the class of threshold mechanisms, which are truthful, very simple in their description, and not (necessarily) affine maximizers. The immediate question is whether there exist other domains for which threshold mechanisms are truthful. We describe here the truthful threshold mechanisms for the translated domain $\lambda D + c$.

Theorem 3. *If D is the domain of additive valuations then a mechanism for the domain $\lambda D + c$ is a threshold mechanism if and only if it satisfies $p_i(a_i, v_{-i}) - c^i_{a_i} = \sum_{j=1}^{m} a_{ij} \big(p_i(\{j\}, v_{-i}) - c^i_{\{j\}}\big)$.*

How to vanish threshold mechanisms. Here we show how starting from the additive domain and slightly enriching the domain of possible valuations we obtain a domain that does not admit any truthful threshold mechanisms. This shows that truthful threshold mechanisms are specific for the domain of additive valuations and its affine transformations and that they cannot be generalized for richer domains.

[1] The proof of Theorem 1 for the 2-player case, goes along exactly the same lines as the proof of Lemma 3.1 [6] by Dobzinski. (The statement of that Lemma involves a different setting, with which we don't deal with in this paper, that of two-player multi-unit auction.)

Let S_i be the set of all valuation functions v_i that are additive. We define the set of valuation functions $S_i+\delta$ as follows: $S_i+\delta$ contains all valuation functions v_i' with $v_i'(a_i) = \sum_{j=1}^{m} a_{ij} v_i(\{j\}) + (|A_i|-1)\delta$ where $\delta \neq 0$ is some constant. That is $v_i \in S_i$ and $v_i' \in S_i+\delta$ agree only on the valuation for getting singletons and the emptyset and differ by some multiple of δ for bigger bundles. Only the sign of δ matters so we can set it to a tiny constant. There exist many other choices of valuations for which our proofs hold. However if you would like in the end to get the characterization of auctions whose valuations satisfy a certain property, say sub-modularity, you should of course mind to make a choice of valuations that are submodular.

We start with two domains, that differ slightly in the valuations one of the players. Each one separately admits truthful threshold mechanisms, but their union does not:

Lemma 4. *Consider a truthful mechanism for the domain $\big(S_1 \cup (S_1+\delta)\big) \times S_2 \times \ldots \times S_2$. If it is a threshold mechanism when restricted to $S_1 \times S_2 \times \ldots \times S_n$, then it is non-threshold when restricted to $(S_1+\delta) \times S_2 \times \ldots \times S_2$.*

Consequently for the domain $\big(S_1 \cup (S_1+\delta)\big) \times S_2 \times \ldots \times S_2$ threshold mechanisms are non-truthful.

Theorem 4. *If the only truthful mechanisms for the domain $S_1 \times S_2 \times \ldots \times S_2$ are either affine maximizers or threshold mechanisms, then the only truthful stable mechanisms, for the domain $\big(S_1 \cup (S_1+\delta)\big) \times S_2 \times \ldots \times S_2$, or any super-domain of it, are affine maximizers.*

Applying our tools for the known characterization. The machinery we just developed opts for a characterization of stable truthful mechanisms for additive combinatorial auctions for n players. We only have one [4] for 2-player mechanisms, that are decisive and allocate all items.

The characterization in [4] is only for additive valuations, applying Theorem 2 it also applies to any affine transformation of the domain of additive valuations. (See the Appendix for the statement of the characterization from [4].) We can now state our main Theorem:

Theorem 5. *The only possible decisive truthful mechanisms for $S_1 \cup (S_1+\delta) \times S_2$ or any super-domain of it are the affine maximizers. 2-player combinatorial auctions that satisfy free disposal, submodularity, subadditivity (or superadditivity) as well as the 2-player unrestricted domain and 2-player combinatorial public projects are some of the super-domains of $S_1 \cup (S_1+\delta) \times S_2$.*

3 Conclusion and future directions

We used as a black box the characterization of 2-player additive combinatorial auctions [4]. This domain is a sub-domain of all domains we mentioned in this work and our results imply that obtaining this characterization is at least as hard as the characterization of all other domains. Observe that by using in Theorem 1 the characterization for n-player subadditive combinatorial auctions in terms of affine maximizers [11] (which assumes stability and scalability but *not* decisiveness) we get that for all super-domains of that domain the only possible mechanisms are the affine maximizers (or similarly using [16] we get a characterization of all superdomains of n-player combinatorial auctions that assumes decisiveness and stability). However these domains do not have 2-player submodular combinatorial auctions as a super-domain. Submodular combinatorial auctions is an important domain [7, 10, 19] whose characterization (assuming decisiveness and that all items are allocated) we obtain in this work for the first time almost for free.

Allthough we characterize at once the very rich class of super-domains of additive combinatorial auctions, the most important aspect of our work is not in characterizing new domains, but in classifying them and obtaining a unified understanding. A more important reason why we used this specific characterization is that it is the only one that involves truthful mechanisms that are not affine maximizers. We enrich the domain very slightly and these mechanisms seize to be truthful, thus the domain of additive combinatorial auctions is the transition domain [19, 15] where the affine maximizers are not any more the only truthful mechanisms. In this way we obtained a classification of many important domains in terms of which domain's characterization we can use as a black box in order to obtain the characterization of all of it's super-domains.

Of course the big open question still remains to obtain characterizations of domains that admit non-stable mechanisms. However the approach of classifying domains in a way similar with the one we propose here provides a more thorough understanding of the existing techniques and results and adds rigor to an intuition that was on the same time helpful and misleading. Can we conjecture that a similar classification holds for the general n-player case?

Acknowledgements: I would like to thank Giorgos Christodoulou, Elias Koutsoupias and Annamária Kovács for helpful discussions and comments.

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A Additional definitions and preliminaries.

An *affine maximizer* is a mechanism defined by a non-negative weight λ_i for each player (at least one of the λ_is is non-zero) and a vector of constants γ where the number of coordinates of the vector is $|A|$. The allocation of an affine maximizer is such that $f(v) \in \text{argmax}_a\{\sum_i \lambda_i v_i + \gamma\}$ and $p_i(v) = -\frac{1}{\lambda_i}\big(\sum_{j\neq i} \lambda_j \cdot v_j + \gamma\big) + h(v_{-i})$.

A mechanism is *decisive* when (for fixed values of the other players) a player can enforce any outcome (allocation), by declaring very high or very low values.

Definition 1 (Threshold mechanism) *A threshold mechanism for the additive combinatorial auctions (/scheduling) domain is one for which there are threshold functions h_{ij} such that the mechanism allocates item j to player i if and only if $v_i(\{j\}) \geq h_{ij}(v_{-i})$. What distinguishes these mechanisms from general mechanisms is that the thresholds depend only on the values of the other players but not on the other values of the player himself. In threshold mechanisms there is a single threshold for getting or not item j and it is the same regardless if the rest of the items allocated to player i.*[2]

We define the *Combinatorial Auction* domain as follows: There is a set of m items for sale and n players/bidders. The alternatives are allocations of the items to bidders.[3] The valuations of the players additionally satisfy $v_i(\emptyset) = 0$ (normalization) and $v_i(A) = v_i(A_i)$ (no externalities). Each item can be allocated to *at most* one player.

This definition is practically the auction that is closest to unrestricted valuations. In the literature [16, 2] the term combinatorial auctions is used for auctions where the valuations of the players also satisfy free disposal. We will however use the term combinatorial auction for the setting defined above and then show that our characterization also holds if we impose any of the following additional restrictions to the valuation function.

Free Disposal: The valuation should be non-decreasing with the set of allocated items, i.e. for every $A_i \subseteq B_i$ we have that $v_i(A_i) \leq v_i(B_i)$. *Sub-additive valuations.* A valuation v_i is subadditive if for any two sets A_i and A'_i, $v_i(A_i) + v_i(A'_i) \geq v_i(A_i \cup A'_i)$. *Superadditive valuations:* For any two disjoint sets A_i and A'_i, $v_i(A_i) + v_i(A'_i) \leq v_i(A_i \cap A'_i)$. *Submodular valuations:* for any two sets A_i and A'_i, $v_i(A_i) + v_i(A'_i) \geq v_i(A_i \cup A'_i) + v_i(A_i \cap A'_i)$. Submodular valuations are a subset of subadditive valuations.

The scheduling domain is essentially the same as the domain of combinatorial auctions with additive valuations. The only difference is that one is a maximization problem and the other minimization. In auctions the utilities of the players are $v - p$ and in scheduling the utilities of the machines are $-v + p$. Therefore we can restate the characterization Theorem from [4] as follows:

Theorem 6 ([4]). *For the combinatorial auction domain D with additive valuations, or any affine transformation of it $\lambda D + c$ the only decisive (decisive for at least 3 outcomes) truthful mechanisms for two players and two items, that allocate all items are either affine maximizers or threshold mechanisms.*

For the case of more than two items every decisive truthful mechanism for 2 players partitions the items into groups. Items in a group of size at least two are allocated by an affine maximizer

[2] It is not true in general that every set of functions h_{ij} defines a legal mechanism, as they have to be consistent between them. In particular, the threshold functions should be such that every item j is allocated to exactly one player. In other words, exactly one of the constraints $v_i(\{j\}) \geq h_{ij}(v_{-i})$, for $i = 1, \ldots, n$, should be satisfied.

[3] We also use the notation a_i for the binary vector where a_{ij} is 1 if player i gets item j and 0 if he doesn't. To go from one notation to the other just consider that $a_{ij} = 1$ if $j \in A_i$ and 0 else.

and items in singleton groups by threshold mechanisms. The allocation of two different groups is not entirely independent: The values of the items in a group allocated by an affine maximizer can appear in the threshold mechanism for a different group of items. The affine maximizers cannot be affected by the values of the items in a group allocated by a threshold mechanism.

Examples of matrix representations of domains with which we deal

– Unrestricted valuations: (each one of the valuations can be any real number)

$$\begin{pmatrix} v_1(a) & v_1(b) & v_1(c) & v_1(d) \\ v_2(a) & v_2(b) & v_2(c) & v_2(d) \\ v_3(a) & v_3(b) & v_3(c) & v_3(d) \end{pmatrix}$$

– Combinatorial public projects: the valuations are *submodular* and $v_i(\emptyset) = 0$. (The valuations are restricted but the outcome is the same for all players just like before.)

$$\begin{pmatrix} v_1(\emptyset) = 0 & v_1(\{1\}) & v_1(\{2\}) & v_1(\{1,2\}) \\ v_2(\emptyset) = 0 & v_2(\{1\}) & v_2(\{2\}) & v_2(\{1,2\}) \\ v_3(\emptyset) = 0 & v_3(\{1\}) & v_3(\{2\}) & v_3(\{1,2\}) \end{pmatrix}$$

– Combinatorial auctions: the valuations are *submodular or subadditive or superadditive or additive* and each item is allocated to exactly one player.

$$\begin{pmatrix} v_1(\emptyset) = 0 & v_1(\{1\}) & v_1(\{2\}) & v_1(\{1,2\}) \\ v_2(\{1,2\}) & v_2(\{2\}) & v_2(\{1\}) & v_2(\emptyset) = 0 \end{pmatrix}$$

– Additive/Scheduling Domain:

$$\begin{pmatrix} v_1(10) & v_1(01) & v_1(10) + v_1(01) & 0 \\ v_2(01) & v_2(10) & 0 & v_2(10) + v_2(01) \end{pmatrix}$$

Setting $a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, d = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ (the names of the alternatives do not matter) we can see that each one of these domains is a subset of the previous domains.

B Missing proofs

Proof (of Lemma 1). (a) Suppose towards a contradiction that there exist v_i, v_i' such that $a_i(v_i, v_{-i}) = a_i(v_i', v_{-i})$, but $p_i(v_i, v_{-i}) < p_i(v_i', v_{-i})$. Then when the true processing times of player i are v_i he has incentive to declare falsely that his processing times are v_i'. His valuation remains the same (as we assumed that $a_i(v_i, v_{-i}) = a_i(v_i', v_{-i})$) and his payment decreases. Consequently by declaring falsely v_i' his utility increases $v_i(a_i(v_i, v_{-i}), v_i) - p_i(v_i, v_{-i}) > v_i(a_i(v_i', v_{-i}), v_i) - p_i(v_i', v_{-i})$. This contradicts the assumption that the mechanism is truthful. (b) Suppose towards a contradiction that there exists a type v such that for some allocation a_i' we have $v_i(a_i(v_i, v_{-i}), v_i) - p_i(a_i(v_i, v_{-i}), v_{-i}) < v_i(a_i', v_i) - p_i(a_i', v_{-i})$. If v_i' is such that $a_i(v_i', v_{-i}) = a_i'$ then player i would have incentive to falsely declare v_i'.

(c) Straightforward from (a) and (b).

Proof (of Lemma 2). Suppose towards a contradiction that there exists some alternative a that is in the range of $f|_{S_1\times\ldots\times V_i\times\ldots\times S_n}(\cdot,v_{-i})$ but not in the range of $f|_{S_1\times\ldots\times S_i\times\ldots S_n}(\cdot,v_{-i})$. (We denote by $f|_D$ the restriction of f to take input only from the domain D.) Take a sufficiently large constant M such that $M >> \max_{a_i}\{p_i(a_i,v_{-i})\}$ and $-M << \min_{a_i}\{p_i(a_i,v_{-i})\}$ define the additive valuation $u_i \in S_i$ such that $u_i(\{j\}) = M$ if $j \in a_i$ and $u_i(\{j\}) = M$ if $j \notin a_i$ (by additivity the valuation for bigger bundles is uniquely determined by the valuation for singletons). For sufficiently large M player i strictly prefers a to all outcomes $b \neq a$. In the mechanism for the bigger domain $S_1 \times \ldots \times V_i \times \ldots S_n$ player i can enforce outcome a by declaring the valuation from V_i for which the mechanism outputs a. (By Lemma 1(a) his payments do no change.)

Proof (of Lemma 3). As the mechanism is truthful for player i and applying Lemma 1(b) its allocation a_i is such that $a_i \in \text{argmax}_{a_i}\{v_i(a_i) - p_i(a_i,v_{-i})\}$ where the payments are of the form $p_i(a_i,v_{-i}) = -\frac{1}{\lambda_i}\big(\sum_{j\neq i}\lambda_j \cdot v_j + \gamma\big)$ (by Lemma 1(a) the payment of player i for the bigger domain only depends on his allocation, thus it is equal to the payment of the restriction of the mechanism to the smaller domain, where the mechanism is an affine maximizer). Plugging in the payments of player i we get $a_i \in \text{argmax}_{a_i}\{v_i(a_i) + \frac{1}{\lambda_i}\big(\sum_{j\neq i}\lambda_j \cdot v_j + \gamma\big)\}$ and consequently $a_i \in \text{argmax}_{a_i}\{\sum_i \lambda_i \cdot v_i + \gamma\}$. This means that the allocation and payments of the first player are the same as that of an affine maximizer.

If there are only 2 players and the allocation of the first player also determines the allocation of the second player, we are done.

If there are more than two players since the mechanism is stable, for every fixed v_{-i}, if the allocation of player i is fixed (and v_i changes), then the allocation of all other players is fixed too. Since for $v_i \in S_i$ the allocation of the other players was that of the affine maximizer $a \in \text{argmax}_a\{\sum_i \lambda_i \cdot v_i + \gamma\}$ it is the same also for $v_i \in V_i$ as long as the allocation of player i remains the same and v_{-i} is fixed. This holds for every fixed v_{-i} and so the mechanism is in whole an affine maximizer, i.e. $a \in \text{argmax}_a\{\sum_i \lambda_i \cdot v_i(a_i) + \gamma\}$.

If $\lambda_i = 0$ then player i is non-decisive and his payment is not required to satisfy any condition. Applying Lemma 2 his range is the same as for the smaller domain, which was a singleton for fixed v_{-i}, so the player remains non-decisive.

Proof (of Theorem 1). The proof is by repeatedly applying Lemma 3 n times, where n is the number of players. We first extend the domain of player 1 from S_1 to V_1, then the domain of player 2 from S_2 to V_2 and so on. That is we consider the following domains in series $S_1 \times \ldots \times S_n$, $V_1 \times S_2 \times \ldots \times S_n$, $V_1 \times V_2 \times S_3 \times \ldots \times S_n$, and so on until we finally extend the domain of the mechanism to $V_1 \times \ldots \times V_n$.

Proof (of Theorem 2). It is easy to see that the allocation part of the mechanism is truthful as it still satisfies the Monotonicity Property.

Take two valuations $v_i \in V_i$ and $v'_i = \lambda_i v_i + c \in \lambda_i V_i + c^i$ we construct a new mechanism that is truthful for $\lambda_i V_i + c^i$ as follows: The allocation part of the two mechanisms for inputs v_i and v'_i respectively is the same so the payments should be such that the boundaries of the mechanism remain the same. Consider the boundaries of the first mechanism between two different allocations a, a'. When the valuations are $v_i \in V_i$ we have (by Lemma 1(c)) $v_i(a_i) - v_i(a'_i) = p(a_i,v_{-i}) - p(a'_i,v_{-i})$. Again by Lemma 1(c) the boundaries of the new mechanism are: $\lambda_i \cdot (v_i(a_i) - v_i(a'_i)) + c_{a_i} - c_{a'_i} = p'(a_i,v_{-i}) - p'(a'_i,v_{-i})$. Combining the latter two equations we get $\lambda_i(p(a_i,v_{-i}) - p(a'_i,v_{-i})) + c_{a_i} - c_{a'_i} = p'(a_i,v_{-i}) - p'(a'_i,v_{-i})$. Setting the payments p'_i of the new mechanism to be $p'_i(a_i,v_{-i}) = \lambda_i \cdot p_i(a_i,v_{-i}) + c_{a_i}$ then they satisfy the previous equation. Consequently starting from a mechanism for $D = V_1 \times \ldots \times V_n$ we got a truthful mechanism for

$V_1 \times \ldots \times \lambda_i V_i + c_i \times \ldots \times V_n$ with the same allocation as the initial mechanism. Repeating the same argument n times we can get a truthful mechanism for the translated domain $\lambda D + c$.

Proof (of Theorem 3). It is easy to show that the payments of a threshold mechanism for the translated domain should satisfy the given condition. Suppose for the other direction that the payments of the mechanism satisfy the given condition. We will show that it is a threshold mechanism. Fix a player i and the values v_{-i} of the other players, for simplicity we will write $p(a)$ instead of $p_i(a_i, v_{-i})$. Take two allocations a and a', that differ only on task k (i.e. $a'_k = 1 - a_k$ and $a_j = a'_j$ for $j \neq k$), then

$$[p(a') - c_{a'}] - [(p(a) - c_a)] = \Big[\sum_{\substack{j=1\\j\neq k}}^{m} a_j \cdot \big(p(\{j\}) - c_{\{j\}}\big) + (1 - a_k)(p(\{k\}) - c_{\{k\}})\Big]$$
$$- \Big[\sum_{\substack{j=1\\j\neq k}}^{m} a_j \big(p(\{j\}) - c_{\{j\}}\big) + a_k(p(\{k\}) - c_{\{k\}})\Big]$$
$$= (1 - 2a_k)(p(\{k\}) - c_{\{k\}}) = (-1)^{a_k}(p(\{k\}) - c_{\{k\}})$$

Since the mechanism is truthful, if a is the allocation produced by the mechanism when the reported valuations are v, it should be $\lambda_i v(a) + c_a - p(a) \geq \lambda_i v(a') + c_{a'} - p(a')$. Taking two allocations a and a', that differ only on task k, rearranging the previous inequality and taking into account that the valuations $v \in D$ are additive, we get $(-1)^{a_k}(\lambda_i v(\{k\}) + c_{\{k\}}) \geq (-1)^{a_k} p(\{k\})$ for every $k = 1, \ldots, m$. Let

$$F_a := \{v(a) \mid (-1)^{a_1}(\lambda_i v(\{1\}) + c_a) \geq (-1)^{a_1} p_i(\{1\}), \ldots,$$
$$(-1)^{a_m}(\lambda_i v(\{m\}) + c_{\{m\}}) \geq (-1)^{a_m} p_i(\{m\})\}$$

Let R_a be the subregion of the input space where the mechanism gives assignment a. These sets satisfy $R_a \subseteq F_a$ and $F_a \cap F_b = \emptyset$, for any two allocations a, b with $a \neq b$. Finally as the mechanism is a partition of the input space, we get that $R_a = F_a$. It is now easy to see that the mechanism is a threshold mechanism: player i gets task k if and only if $\lambda_i v_i(\{k\}) + c_{\{k\}} \geq p_i(\{k\}, t_{-i})$.

Proof (of Lemma 4). If a mechanism is truthful for $\big(S_1 \cup (S_1 + \delta)\big) \times S_2 \times \ldots \times S_n$, then by Lemma 1(a) the payments of player 1 are the same regardless of whether his valuation is from S_1 or from $S_1 + \delta$. From Theorem 3 if a mechanism is a threshold mechanism (for a subset of items) for the domain of valuations S_1 it satisfies $p_1(a_1, v_{-1}) = \sum_{j=1}^{m} a_{1j} p_1(\{j\}, v_{-1})$. Supposing towards a contradiction that it is a threshold mechanism (for the same or a smaller subset of items) also for $S_1 + \delta$, it would, again by Theorem 3, satisfy $p_1(a_1, v_{-1}) - \delta(|A_1| - 1) = \sum_{j=1}^{m} a_{1j} p_1(\{j\}, v_{-1})$. Combining the latter two relations, and since $\delta \neq 0$, we get that the mechanisms cannot be a threshold mechanism for any subset of two or more items when it is restricted to $S_1 \times S_2 \times \ldots \times S_n$.

Proof (of Theorem 4). By Theorem 2 since we assumed that the only truthful mechanisms for $S_1 \times S_2 \times \ldots \times S_2$ are either affine maximizers or threshold mechanisms the same characterization also holds for $(S_1 + \delta) \times S_2 \times \ldots \times S_2$. By Lemma 4 a threshold mechanism for $S_1 \times S_2 \times \ldots \times S_n$ is non-threshold for $(S_1 + \delta) \times S_2 \times \ldots \times S_n$, so by our assumption it can be nothing else than an affine maximizer for $(S_1 + \delta) \times S_2 \times \ldots \times S_n$. Applying Lemma 3 the mechanisms is also an affine maximizer when the domain of player 1 is enlarged, so the only possible stable mechanisms for $\big(S_1 \cup (S_1 + \delta)\big) \times S_2 \times \ldots \times S_n$ are the affine maximizers. By Theorem 1 the same characterization also holds for any super-domain.

Proof (of Theorem 5). Combining Theorems 6 and 4 we get that the only truthful mechanisms for the domain $\big(S_1 \cup (S_1+\delta)\big) \times S_2$ are affine maximizers. By Theorem 1 the characterization also holds for any super-domain of it.

For appropriate choice of the sign of δ we have that the domain $\big(S_1 \cup (S_1+\delta)\big) \times S_2$ is a sub-domain of superadditive, subadditive or submodular combinatorial auctions as well as of public projects or the unrestricted domain. By Theorem 4(b) and and Theorem 6 the only decisive mechanisms for $\big(S_1 \cup (S_1+\delta)\big) \times S_2$ are the affine maximizers and by Theorem 1 the same holds for any super-domain of $\big(S_1 \cup (S_1+\delta)\big) \times S_2$.

Obviously additive valuation functions satisfy all these conditions. Valuation functions from $S_1+\delta$ satisfy for $\delta > 0$ superadditivity and free disposal and for $\delta < 0$ subadditivity and submodularity.

C A note on presenting different domains of valuations as restrictions of the combinatorial auction domain.

Any domain is a subset of $\mathbb{R}^{|\mathcal{A}|}$, where $|\mathcal{A}|$ is the number of possible outcomes. Here we will generalize a representation of the domain of valuations that lead to interesting results for the scheduling problem [5, 4, 21].

Say there are only two items, then the possible outcomes are the four allocations $\{1,2\},\{1\},\{2\},\emptyset$. The natural way to represent the domain of valuations of a player i is to have one axis for the valuation of the player for each one of the outcomes (except for $v_i(\emptyset)$, which, assuming that the domain is normalized, only takes one possible value $v_i(\emptyset)=0$). So on the x-axis we have the valuation $v_i(\{1\})$ of player i for getting only the first item on the y-axis we have the valuation $v_i(\{2\})$ of the same player for getting only the second item and finally on the z-axis his valuation $v_i(\{1,2\})$ for getting both items.

Basically all auction domains we are interested in are different restrictions of this domain. Restrictions in which we are usually interested in, is to assume additivity of the valuations (our domain is only the plane $z=x+y$), or sub-additivity (our domain is the subset of $\mathbb{R}^3$ satisfying $z \geq x+y$), or free disposal (i.e. $z \geq x$ and $z \geq y$). For example the scheduling domain assumes additive valuations. Finally multi-unit (here two-unit) combinatorial auctions are the subset of $\mathbb{R}^3$ where $x=y$ and $z \geq x$, this is basically the only important domain whose characterization we won't get from scheduling for the simple reason that the additive combinatorial auction domain is not its subdomain.