

Auctions with Heterogeneous Items and Budget Limits*

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Abstract

We study individual rational, Pareto optimal, and incentive compatible mechanisms for auctions with heterogeneous items and budget limits. For multi-dimensional valuations we show that there can be no *deterministic* mechanism with these properties for *divisible* items. We use this to show that there can also be no *randomized* mechanism that achieves this for either *divisible* or *indivisible* items. For single-dimensional valuations we show that there can be no *deterministic* mechanism with these properties for indivisible items, but that there is a *randomized* mechanism that achieves this for either divisible or indivisible items. The impossibility results hold for *public* budgets, while the mechanism allows *private* budgets, which is in both cases the harder variant to show. While all positive results are polynomial-time algorithms, all negative results hold independent of complexity considerations.

1 Introduction

A canonical problem in Mechanism Design is the design of economically efficient auctions that satisfy individual rationality and incentive compatibility. In settings with quasi-linear utilities these goals are achieved by the Vickrey-Clarke-Groves (VCG) mechanism. In many practical situations, including settings in which the agents have budget limits, the quasi-linear assumption fails to be true and, thus, the VCG mechanism is not applicable.

Ausubel [2] describes an ascending-bid auction for homogeneous items that yields the same outcome as the sealed-bid Vickrey auction, but offers advantages in terms of simplicity, transparency, and privacy preservation. In his concluding remarks he points out that “when budgets impair the bidding of true valuations in a sealed-bid Vickrey auction, a dynamic auction may facilitate the expression of true valuations while staying within budget limits” (p. 1469).

Dobzinski et al. [7] show that an adaptive version of Ausubel’s “clinging auction” is indeed the unique mechanism that satisfies individual rationality, Pareto optimality, and

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incentive compatibility in settings with *public* budgets. They use this fact to show that there can be no mechanism that achieves those properties for *private* budgets.

An important restriction of Dobzinski et al.’s impossibility result for private budgets is that it only applies to *deterministic* mechanisms. In fact, as Bhattacharya et al. [4] show, there exists a *randomized* auction that is individual rational, Pareto optimal, and incentive compatible with private budgets.

All these results assume that the items are homogeneous, although as Ausubel [3] points out, “situations abound in diverse industries in which heterogeneous (but related) commodities are auctioned” (p. 602). He also describes an ascending-bid auction, the “crediting and debiting auction”, that takes the place of the “clinching auction” when items are heterogeneous.

Positive and negative results for *deterministic* mechanisms and public budgets are given in [8, 10, 9, 6]. We focus on *randomized* mechanisms, and prove positive results for private budgets and negative results for public budgets. *We thus explore the power and limitations of randomization in settings with heterogeneous items and budget limits.*

Model. There are n agents and m items. The items are either divisible or indivisible. Each agent has a valuation for each item and each agent has a budget. Agents can be assigned more than one item and valuations are additive across items. All valuations are private. We distinguish between settings in which budgets are public and settings in which budgets are private. A mechanism is used to compute assignments and payments based on the reported valuations and the reported budgets. An agent’s utility is defined as valuation for the assigned items minus the payment if the payment does not exceed the budget and the utility is minus infinity otherwise. We assume that agents are utility maximizers and as such need not report their true valuations and true budgets.

Our goal is to design mechanisms with certain desirable properties or to show that no such mechanism exists. For deterministic mechanisms we require that the respective properties are always satisfied. For randomized mechanisms we either require that the properties hold for all outcomes or that they hold in expectation. In the former case they are satisfied *ex post*, in the latter they are satisfied *ex interim*.

We are interested in the following properties:

(a) *Individual rationality (IR)*: A mechanism is IR if all outcomes it produces give non-negative utility to the agents *and* the sum of the payments is non-negative. (b) *Pareto optimality (PO)*: A mechanism is PO if it produces an outcome such that there is no other outcome in which all agents and the auctioneer are no worse off and at least one of the agents or the auctioneer is strictly better off.¹ (c) *No positive transfers (NPT)*: A mechanism satisfies NPT if it produces an outcome in which all payments are non-negative.

¹If the outcome for which we want to establish PO is IR, then we only have to consider alternative outcomes that are IR. In the alternative outcome individual payments may be negative, even if the original outcome satisfied IR and NPT. See the arXiv version of [8] for a more detailed discussion.

(d) *Incentive compatibility (IC)*: A mechanism is IC if each agent maximizes his utility by reporting his true valuation(s) and true budget no matter what the other agents' reported valuations and reported budgets are. If the budget is public then the agents can only report their true budgets.

Following prior work we focus on IR, PO, NPT, and IC for positive results and on IR, PO, and IC for negative results. Both the inclusion of NPT for positive results and the exclusion of NPT for negative results strengthens the respective results.

Results. We analyze two settings with heterogeneous items, one with multi-dimensional valuations and one with single-dimensional valuations. In the setting with multi-dimensional valuations, each agent has an arbitrary, non-negative valuation for each of the items. In the setting with single-dimensional valuations, which is inspired by sponsored search auctions, an agent's valuation for an item is the product of an item-specific quality and an agent-specific valuation. Our motivation for studying this setting is that an advertiser might want to show his ad in multiple slots on a search result page.

(a) For **multi-dimensional** valuations the impossibility result of [8] implies that there can be no deterministic mechanism for *indivisible* items that is IR, PO, and IC for public budgets. We show that there also can be no deterministic mechanism with these properties for *divisible* items. We use this to show that for both divisible and indivisible items there can be no randomized mechanism that is IR ex interim, PO ex interim, and IC ex interim. This is the first impossibility result for randomized mechanisms for auctions with budget limits. It establishes an interesting separation between randomized mechanisms for single-dimensional valuations, where such mechanisms exist (see below), and multi-dimensional valuations, where no such mechanism exists.

(b) For **single-dimensional** valuations the impossibility result of [7] implies that there can be no deterministic mechanism for indivisible items that is IR, PO, and IC for *private* budgets. We show that for heterogeneous items there can also be no deterministic mechanism for indivisible items that is IR, PO, and IC for *public* budgets. We thus obtain a strong separation between deterministic mechanisms, that do *not* exist for *public* budgets, and randomized mechanisms, that exist for *private* budgets (see below). This separation is stronger than in the homogeneous items setting, where a deterministic mechanism exists for public budgets [7]. Additionally, our impossibility result is tight in the sense that if any of the conditions is relaxed such a mechanism exists: (i) For *homogeneous*, indivisible items a deterministic mechanism is given in [7], (ii) we give a deterministic mechanism for heterogeneous, *divisible* items, and (iii) we give a *randomized* mechanism for heterogeneous, indivisible items.

(c) For **single-dimensional** valuations we give mechanisms that extend earlier work for homogeneous items to heterogeneous items. Specifically, we give a *randomized* mechanism that satisfies IR ex interim, NPT ex post, PO ex post, and IC ex interim for *divisible* or *indivisible* items and *public* or *private* budgets. Additionally, for the case of *divisible* items and *public* budgets we give a *deterministic* mechanism that is IR, NPT, PO, and IC.

We summarize our results and the results from related work described next in Table 1 and Table 1 below.

Related Work. The setting in which all items are identical was first studied by [7]. By adapting the “clinch auction” of [2] from settings without budgets to settings with budgets they obtain deterministic mechanisms that are IR, NPT, PO, and IC with public budgets for divisible and indivisible items. They also show that these mechanisms are the only mechanisms that are IR, PO, and IC, and that they are not IC for private budgets, implying that there can be no deterministic mechanism that is IR, PO, and IC when the budgets are private. However, [4] showed that there is such a mechanism for private budgets that is randomized. Note that both, [7] and [4] study only homogeneous items.

Impossibility results for general, non-additive valuations were given in [10, 6, 9]. Combined they show that there can be no deterministic mechanism for indivisible items that is IR, PO, and IC with public budgets for monotone valuations with decreasing marginals. These impossibility results do not apply to additive valuations, which is the case that we study.

Heterogeneous items were first studied in [8]. In their model each agent has the same valuation for each item in an agent-dependent interest set and zero for all other items. They give a deterministic mechanism for indivisible items that satisfies IR, NPT, PO, and IC when both interest sets and budgets are public. They also show that when the interest sets are *private*, then there can be no deterministic mechanism that satisfies IR, PO, and IC. This implies that for *indivisible* items and public budgets there can be no *deterministic* IR, PO, and IC mechanism for unconstrained valuations.

Settings with heterogeneous items were in parallel to this paper studied by [6] and [9]. The former study problems with multiple keywords, each having multiple slots. Agents have unit demand per keyword. They are either interested in a subset of the keywords and have identical valuations for the slots or they are interested in all keywords and have sponsored search like valuations for the slots. The latter study settings in which the agents have identical valuations and the allocations must satisfy polymatroidal or polyhedral constraints.

The settings studied in [6, 9] are more general than the single-dimensional valuations setting studied here. On the one hand this implies that their positive results apply to the single-dimensional valuations setting studied here, and show that there are deterministic mechanisms for divisible items and randomized mechanisms for both divisible and indivisible items that are IC with *public* budgets. On the other hand this implies that our negative result for the single-dimensional valuations setting applies to the settings studied in these papers, and shows that there can be no *deterministic* mechanisms that are IC with *public* budgets for *indivisible* items. Finally, the impossibility results presented in [6, 9] either assume that the valuations are non-additive or that the allocations satisfy arbitrary polyhedral constraints and have therefore no implications for the multi-dimensional valuations setting studied here.

Overview. We summarize the results from related work and this paper for indivisible items in Table 1 and for divisible items in Table 1. We use a plus (+ or \oplus) to indicate that there is an IR, PO, NPT, and IC mechanism. We use a minus ($-$ or \ominus) to indicate that there is no IR, PO, and IC mechanism. We use + and $-$ for results from related work and \oplus and \ominus for results from this paper. A question mark (?) indicates that nothing is known for this setting. For the model of [8] the table has two entries, one for public and one for private interest sets. *While all positive results from this paper are polynomial-time algorithms, all negative results hold independent of complexity considerations.*

Table 1: Results for *Indivisible* Items from Related Work and this Paper

		homogeneous		interest set	heterogeneous & additive		
budgets		add.	non-add.	public/private	multi-keyword unit demand	single-dim.	multi-dim.
det.	public	+ [7]	-[10, 6]	+ [8]/-[8]	\ominus	\ominus	- [8]
	private	- [7]	- [7]	-[7]/-[7]	- [7]	-[7]	- [7]
rand.	public	+ [7]	?	+ [8]/?	+ [6, 9]	\oplus	\ominus
	private	+ [4]	?	?/?	?	\oplus	\ominus

Table 2: Results for *Divisible* Items from Related Work and this Paper

		homogeneous		polymatroid	heterogeneous & additive		
budgets		add.	non-add.	constraints	multi-keyword unit demand	single-dim.	multi-dim.
det.	public	+ [7, 4]	-[9]	+ [9]	+ [6, 9]	\oplus	\ominus
	private	- [7]	-[7]	- [7]	- [7]	- [7]	- [7]
rand.	public	+ [7, 4]	?	+ [9]	+ [6, 9]	\oplus	\ominus
	private	+ [4]	?	?	?	\oplus	\ominus

Techniques. Our technical contributions are as follows:

(a) For multi-dimensional valuations we obtain a partial characterization of IC by generalizing the “weak monotonicity” (WMON) condition of [5] from settings *without budgets* to settings *with public budgets*. We obtain our impossibility result for deterministic mechanisms and divisible items by showing that in certain settings WMON will be violated. For this we use that multi-dimensional valuations enable the agents to lie in a sophisticated way: While all previous impossibility proofs in this area used agents that either only overstate or only understate their valuations, we use an agent that overstates his valuation for one item and understates his valuation for another.

(b) For single-dimensional valuations and both divisible and indivisible items we characterize PO by a simpler “no trade” (NT) condition. Although this condition is more complex than similar conditions in [7, 4, 8], we are able to show that an outcome is PO if and only if it satisfies NT. We also generalize the “classic” characterization results of IC mechanism of [11, 1] from settings *without budgets* to settings *with public budgets* by

showing that a mechanism is IC with public budgets if and only if it satisfies “value monotonicity” (VM) and “payment identity” (PI). The characterizations of PO and IC with public budgets play a crucial role in the proof of our impossibility result for indivisible items, which uses NT and PI to derive lower bounds on the agents’ payments that conflict with the upper bounds on the payments induced by IR.

(c) We establish the positive results for single-dimensional valuations and both divisible and indivisible items by giving a new reduction of this case to the case of a single and by definition homogeneous item. This allows us to apply the techniques that [4] developed for the single-item setting. This is a general reduction between the heterogeneous items setting and the homogeneous items setting, which is likely to have further applications.

(d) We give an explicit polynomial-time algorithm for the “adaptive clinching auction” for *divisible* items and an arbitrary number of agents. To the best of our knowledge we are the first ones to actually give a polynomial-time version of this auction for arbitrarily many agents.

2 Problem Statement

We are given a set N of n agents and a set M of m items. We distinguish between settings with divisible items and settings with indivisible items. In both settings we use $X = \prod_{i=1}^n X_i$ for the allocation space. For divisible items the allocation space is $X_i = [0, 1]^m$ for all agents $i \in N$ and $x_{i,j} \in [0, 1]$ denotes the fraction of item $j \in M$ that is allocated to agent $i \in N$. For indivisible items the allocation space is $X_i = \{0, 1\}^m$ for all agents $i \in N$ and $x_{i,j} \in \{0, 1\}$ indicates whether item $j \in M$ is allocated to agent $i \in N$ or not. In both cases we require that $\sum_{i=1}^n x_{i,j} \leq 1$ for all items $j \in M$. We do *not* require that $\sum_{j=1}^m x_{i,j} \leq 1$ for all agents $i \in N$, i.e., we do *not* assume that the agents have unit demand.

Each agent i has a type $\theta_i = (v_i, b_i)$ consisting of a valuation function $v_i : X_i \rightarrow \mathbb{R}_{\geq 0}$ and a budget $b_i \in \mathbb{R}_{\geq 0}$. We use $\Theta = \prod_{i=1}^n \Theta_i$ for the type space. We consider two settings with heterogeneous items, one with multi- and one with single-dimensional valuations. In the first setting, each agent $i \in N$ has a valuation $v_{i,j} \in \mathbb{R}_{\geq 0}$ for each item $j \in M$ and agent i ’s valuation for allocation x_i is $v_i(x_i) = \sum_{j=1}^m x_{i,j} v_{i,j}$. In the second setting, which is inspired by sponsored search auctions, each agent $i \in N$ has a valuation $v_i \in \mathbb{R}_{\geq 0}$, each item $j \in M$ has a quality $\alpha_j \in \mathbb{R}_{\geq 0}$, and agent i ’s valuation for allocation $x_i \in X_i$ is $v_i(x_i) = \sum_{j=1}^m x_{i,j} \alpha_j v_i$. For simplicity we will assume that in this setting $\alpha_1 > \alpha_2 > \dots > \alpha_m$ and that $v_1 > v_2 > \dots > v_n > 0$.

A (direct revelation) mechanisms $M = (x, p)$ consisting of an allocation rule $x : \Theta \rightarrow X$ and a payment rule $p : \Theta \rightarrow \mathbb{R}^n$ is deployed to compute an outcome (x, p) consisting of an allocation $x \in X$ and payments $p \in \mathbb{R}^n$. We say that a mechanism is deterministic if the computation of (x, p) is deterministic, and it is randomized if the computation of (x, p) is randomized.

We assume that the agents are utility maximizers and as such need not report their types truthfully. We consider settings in which both the valuations and budgets are private and settings in which only the valuations are private and the budgets are public. When the valuations resp. budgets are private, then the other agents have no knowledge about them, not even about their distribution. In the former setting a report by agent $i \in N$ with true type $\theta_i = (v_i, b_i)$ can be any type $\theta'_i = (v'_i, b'_i)$. In the latter setting agent $i \in N$ is restricted to reports of the form $\theta'_i = (v'_i, b_i)$. In both settings, if mechanism $M = (x, p)$ is used to compute an outcome for reported types $\theta' = (\theta'_1, \dots, \theta'_n)$ and the true types are $\theta = (\theta_1, \dots, \theta_n)$ then the utility of agent $i \in N$ is

$$u_i(x_i(\theta'), p_i(\theta'), \theta_i) = \begin{cases} v_i(x_i(\theta')) - p_i(\theta') & \text{if } p_i(\theta') \leq b_i, \text{ and} \\ -\infty & \text{otherwise.} \end{cases}$$

For deterministic mechanisms and their outcomes we are interested in the following properties:

(a) *Individual rationality (IR)*: A mechanism is IR if it always produces an IR outcome. An outcome (x, p) for types $\theta = (v, b)$ is IR if it is (i) *agent rational*: $u_i(x_i, p_i, \theta_i) \geq 0$ for all agents $i \in N$ and (ii) *auctioneer rational*: $\sum_{i=1}^n p_i \geq 0$. (b) *Pareto optimality (PO)*: A mechanism is PO if it always produces a PO outcome. An outcome (x, p) for types $\theta = (v, b)$ is PO if there is no other outcome (x', p') such that $u_i(x'_i, p'_i, \theta_i) \geq u_i(x_i, p_i, \theta_i)$ for all agents $i \in N$ and $\sum_{i=1}^n p'_i \geq \sum_{i=1}^n p_i$, with at least one of the inequalities strict.² (c) *No positive transfers (NPT)*: A mechanism satisfies NPT if it always produces an NPT outcome. An outcome (x, p) satisfies NPT if $p_i \geq 0$ for all agents $i \in N$. (d) *Incentive compatibility (IC)*: A mechanism satisfies IC if for all agents $i \in N$, all true types θ , and all reported types θ' we have $u_i(x_i(\theta_i, \theta'_{-i}), p_i(\theta_i, \theta'_{-i}), \theta_i) \geq u_i(x_i(\theta'_i, \theta'_{-i}), p_i(\theta'_i, \theta'_{-i}), \theta_i)$.

If a randomized mechanism satisfies any of these conditions in expectation, then we say that the respective property is satisfied *ex interim*. If it satisfies any of these properties for all outcomes it produces, then we say that it satisfies the respective property *ex post*.

3 Multi-Dimensional Valuations

In this section we obtain a partial characterization of mechanisms that are IC with public budgets by generalizing the “weak monotonicity” condition of [5] from settings without budgets to settings with budgets. We use this partial characterization together with a sophisticated way of lying, in which an agent understates his valuation for some item and overstates his valuation for another item, to prove that there can be no *deterministic* mechanism for *divisible* items that is IR, PO, and IC with public budgets. Afterwards, we use this result to show that there can be no *randomized* mechanism for either *divisible* or *indivisible* items that is IR ex interim, PO ex interim, and IC ex interim for public budgets.

²Both IR and PO are defined with respect to the reported types, and are satisfied with respect to the true types only if the mechanism also satisfies IC.

Partial Characterization of IC. For settings *without budgets* every mechanism that is incentive compatible must satisfy what is known as *weak monotonicity (WMON)*, namely if x'_i and x_i are the assignments of agent i for reports v'_i and v_i , then the difference in the valuations for the two assignments must be at least as large under v'_i as under v_i , i.e., $v'_i(x_i(\theta'_i, \theta_{-i})) - v'_i(x_i(\theta_i, \theta_{-i})) \geq v_i(x_i(\theta'_i, \theta_{-i})) - v_i(x_i(\theta_i, \theta_{-i}))$. We show that this is also true for mechanisms that respect the publicly known budget limits.³

Proposition 1. *If a mechanism $M = (x, p)$ for multi-dimensional valuations and either divisible or indivisible items that respects the publicly known budget limits is IC, then it satisfies WMON.*

Deterministic Mechanisms for Divisible Items. We prove the impossibility result by analyzing a setting with two agents and two items. This restriction is without loss of generality as the impossibility result for an arbitrary number of agents $n > 2$ and an arbitrary number of items $m > 2$ follows by setting $v_{i,j} = 0$ if $i > 2$ or $j > 2$. In our impossibility proof agent 2 is not budget restricted (i.e., $b_2 > v_{2,1} + v_{2,2}$). Agents can lie when they report their valuations, and it is not sufficient to study a single input to prove the impossibility. Hence, we study the outcome for three related cases, namely Case 1 where $v_{1,1} < v_{2,1}$ and $v_{1,2} < v_{2,2}$; Case 2 where $v_{1,1} > v_{2,1}$, $v_{1,2} < v_{2,2}$, and $b_1 > v_{1,1}$; and Case 3 where $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, and additionally, $b_1 > v_{1,1}$, $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, and $v_{2,1} + v_{2,2} > b_1$. We give a partial characterization of those cases, which allows us to analyze the rational behavior of the agents.

Case 1 is easy: Agent 2 is not budget restricted and has the highest valuations for both items; so he will get both items. Thus the utility for agent 1 is zero. Based on this observation Case 2 can be analyzed: Agent 1 has the higher valuation for item 1, while agent 2 has the higher valuation for item 2. Thus, agent 1 gets item 1 and agent 2 gets item 2. Since the only difference to Case 1 is that in Case 2 $v_{1,1} > v_{2,1}$ while in Case 1 $v_{1,1} < v_{2,1}$, the critical value whether agent 2 gets item 1 or not is $v_{2,1}$. Thus, in every IC mechanism, agent 1 has to pay $v_{2,1}$ and has utility $v_{1,1} - v_{2,1}$. The details of these proofs can be found in Appendix B. Using these observations we are able to exactly characterize the allocation produced in Case 3 as follows: In Case 3 agent 1 has a higher valuation than agent 2 for both items, but he does not have enough budget to pay for both fully. First we show that if agent 1 does not spend his whole budget ($p_1 < b_1$) he must fully receive both items (specifically $x_{1,2} = 1$), since if not, he would buy more of them. Additionally, even if he spent his budget fully (i.e., $p_1 = b_1$) his utility u_i , which equals $x_{1,1}v_{1,1} + x_{1,2}v_{1,2} - b_1$, must be non-negative. Since $b_1 > v_{1,1}$ this implies that $x_{1,1}$ must be 1, i.e., he must receive item 1 fully, and $x_{1,2}$ must be non-zero.

³Without this restriction we could charge $p_i > b_i$ from all agents $i \in N$ to be IC. This restriction is satisfied by IR mechanisms to which we will apply this result.

Lemma 1. *Given $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, $b_1 > v_{1,1}$, and $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, if $p_1 < b_1$ then $x_{1,1} = 1$ and $x_{1,2} = 1$, else if $p_1 = b_1$ then $x_{1,1} = 1$ and $x_{1,2} > 0$, in every IR and PO outcome.*

Then we show that actually $x_{1,2} < 1$, which, combined with the previous lemma, implies that $p_1 = b_1$. The fact that $x_{1,2} < 1$, i.e, that agent 1 does not fully get item 1 and 2 is not surprising since he does not have enough budget to outbid agent 2 on both items as $b_1 < v_{2,1} + v_{2,2}$. However, we are even able to determine the exact value of $x_{1,2}$, which is $(b_1 - v_{2,1})/v_{2,2}$.

Lemma 2. *Given $b_2 > v_{2,1} + v_{2,2}$, $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, $b_1 > v_{1,1}$, $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, and $v_{2,1} + v_{2,2} > b_1$, then $p_1 = b_1$ and $x_{1,2} = (b_1 - v_{2,1})/v_{2,2} < 1$ in every IR and PO outcome selected by an IC mechanism.*

We combine these characterizations of Case 3 with (a) the WMON property shown in Proposition 1 and (b) a sophisticated way of the agent to lie: He *overstates* his value for item 1 by a value α and *understates* his value for item 2 by a value $0 < \beta < \alpha$, but by such small values that Case 3 continues to hold. Thus, by Lemma 1 $x_{2,1}$ remains 0 (whether the agent lies or does not), and thus, the WMON condition implies that $x_{2,2}$ does *not* increase. However, by the dependence of $x_{1,2}$ on $v_{2,1}$ and $v_{2,2}$ shown in Lemma 2, $x_{1,2}$, and thus also $x_{2,2}$ changes when agent 2 lies. This gives a contradiction to the assumption that such a mechanism exists.

Theorem 1. *There is no deterministic IC mechanism for divisible items which selects for any given input with public budgets an IR and PO outcome.*

Proof. Let us assume by contradiction that such a mechanism exists and consider an input for which $b_2 > v_{2,1} + v_{2,2}$, $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, $b_1 > v_{1,1}$, $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, and $v_{2,1} + v_{2,2} > b_1$ holds. Such an input exists, for example $v_{1,1} = 4$, $v_{1,2} = 5$, $v_{2,1} = 3$, and $v_{2,2} = 4$ with budgets $b_1 = 5$ and $b_2 = 8$ would be such an input. Lemma 1 and 2 imply that $x_{1,1} = 1$, $x_{2,1} = 0$, $x_{1,2} = \frac{b_1 - v_{2,1}}{v_{2,2}}$, $x_{2,2} = 1 - x_{1,2}$, and $p_1 = b_1$. Let us consider an alternative valuation by agent 2. We define $v'_{2,1} = v_{2,1} + \alpha$ and $v'_{2,2} = v_{2,2} - \beta$ for arbitrary $\alpha, \beta > 0$ and $\alpha > \beta$ which are sufficiently small such that $v_{1,1}v'_{2,2} > v_{1,2}v'_{2,1}$ holds. By Proposition 1, IC implies WMON, and therefore, $x'_{2,2}v'_{2,2} - x_{2,2}v'_{2,2} \geq x'_{2,2}v_{2,2} - x_{2,2}v_{2,2}$. It follows that $x_{2,2} \geq x'_{2,2}$, and by Lemma 2, $\frac{b_1 - v_{2,1}}{v_{2,2}} \leq \frac{b_1 - v'_{2,1}}{v'_{2,2}}$. Hence, the budget of agent 1 has to be large enough, such that $b_1 \geq \frac{v_{2,2}v'_{2,1} - v_{2,1}v'_{2,2}}{v_{2,2} - v'_{2,2}} = \frac{v_{2,1}\beta + v_{2,2}\alpha}{\beta} > v_{2,1} + v_{2,2}$, but $b_1 < v_{2,1} + v_{2,2}$ holds by assumption. Contradiction! \square

Randomized Mechanisms for Divisible and Indivisible Items. We exploit the fact that randomized mechanisms for both divisible and indivisible items are essentially equivalent to deterministic mechanisms for divisible items.

We show that for agents with budget constraints every randomized mechanism $\bar{M} = (\bar{x}, \bar{p})$ for divisible or indivisible items can be mapped bidirectionally to a deterministic mechanism $M = (x, p)$ for divisible items with identical expected utility for all the agents and the auctioneer when the same reported types are used as input. To turn a randomized mechanism for *indivisible* items into a deterministic mechanism for *divisible* items simply compute the expected values of p_i and $x_{i,j}$ for all i and j and return them. To turn a deterministic mechanism for *divisible* items into a randomized mechanism for *indivisible* items simply pick values with probability $x_{i,j}$ and keep the same payment as the deterministic mechanism.

Proposition 2. *Every randomized mechanism $\bar{M} = (\bar{x}, \bar{p})$ for agents with finite budgets, a rational auctioneer, and a limited amount of divisible or indivisible items can be mapped bidirectionally to a deterministic mechanism $M = (x, p)$ for divisible items such that $u_i(x_i(\theta'), p_i(\theta'), \theta_i) = \mathbb{E}[u_i(\bar{x}_i(\theta'), \bar{p}_i(\theta'), \theta_i)]$ and $\sum_{i \in N} p_i(\theta') = \mathbb{E}[\sum_{i \in N} \bar{p}_i(\theta')]$ for all agents i , all true types $\theta = (v, b)$, and reported types $\theta' = (v', b')$.*

Proof. Let us map $\bar{M} = (\bar{x}, \bar{p})$ to $M = (x, p)$ that assigns for each agent $i \in N$ and item $j \in M$ a fraction of $\mathbb{E}[\bar{x}_{i,j}]$ of item j to agent i , and makes each agent $i \in N$ pay $\mathbb{E}[\bar{p}_i]$. The expectations exist since the feasible fractions of items and the feasible payments have an upper bound and a lower bound. For the other direction, we map $M = (x, p)$ to $\bar{M} = (\bar{x}, \bar{p})$ that randomly picks for each item $j \in M$ an agent $i \in N$ to which it assigns item j in a way such that agent i is picked with probability $x_{i,j}$, and makes each agent $i \in N$ pay p_i . Since $x = \mathbb{E}[\bar{x}]$ and $p = \mathbb{E}[\bar{p}]$, $\sum_{j \in M} (x_{i,j} v_{i,j}) - p_i = \mathbb{E}[\sum_{j \in M} (\bar{x}_{i,j} v_{i,j}) - \bar{p}_i]$ for all $i \in N$ and $\sum_{i \in N} p_i = \mathbb{E}[\sum_{i \in N} \bar{p}_i]$. \square

This proposition implies the non-existence of randomized mechanisms stated in Theorem 2.

Theorem 2. *There can be no randomized mechanism for divisible or indivisible items that is IR ex interim, PO ex interim, and IC ex interim, and that satisfies the public budget constraint ex post.*

Proof. For a contradiction suppose that there is such a randomized mechanism. Then, by Proposition 2, there must be a deterministic mechanism for divisible items and public budgets that satisfies IR, PO, and IC. This gives a contradiction to Theorem 1. \square

4 Single-Dimensional Valuations

In this section we present exact characterizations of PO outcomes and mechanisms that are IC with public budgets. We characterize PO by a simpler “no trade” condition and, similar to Section 3, we extend the “classic” characterization results for IC mechanisms for single-dimensional valuations (see, e.g., [11, 1]) without budgets to settings with public budgets.

We use these characterizations to show that there can be no deterministic mechanism for divisible items that is IR, PO, and IC with public budgets. We also present a reduction to the setting with a single (and thus homogeneous) item that allows us to apply the following proposition from [4]. The basic building block of the mechanisms mentioned in this proposition is the “adaptive clinching auction” for a single divisible item. It is described for two agents in [7], as a “continuous time process” for arbitrarily many agents in [4], and as an explicit polynomial-time algorithm for arbitrarily many agents in Appendix E.

Proposition 3 ([4]). *For a single divisible item there exists a deterministic mechanism that satisfies IR, NPT, PO, and IC for public budgets. Additionally, for a single divisible or indivisible item there exists a randomized mechanism that satisfies IR ex interim, NPT ex post, PO ex post, and IC ex interim for private budgets.*

Exact Characterizations of PO and IC. We start by characterizing PO outcomes through a simpler “no trade” condition. Outcome (x, p) for single-dimensional valuations and either divisible or indivisible items that respects the budget limits satisfies *no trade* (NT) if (a) $\sum_{i \in N} x_{i,j} = 1$ for all $j \in M$, and (b) there is no x' such that for $\delta_i = \sum_{j \in M} (x'_{i,j} - x_{i,j}) \alpha_j$ for all $i \in N$, $W = \{i \in N \mid \delta_i > 0\}$, and $L = \{i \in N \mid \delta_i \leq 0\}$ we have $\sum_{i \in N} \delta_i v_i > 0$ and $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq 0$.⁴ This definition says that there should be no alternative assignment that overall increases the sum of the valuations, and allows the “winners” to compensate the “losers”. It differs from the definitions in prior work in that it allows trades that involve both items *and* money. We will exploit this fact in the proof of our impossibility result.

Proposition 4. *Outcome (x, p) for single-dimensional valuations and either divisible or indivisible items that respects the budget limits is PO if and only if it satisfies NT.*

Next we characterize mechanisms that are IC with public budgets by “value monotonicity” and “payment identity”. Mechanism $M = (x, p)$ for single-dimensional valuations and indivisible items that respects the publicly known budgets satisfies *value monotonicity* (VM) if for all $i \in N$, $\theta_i = (v_i, b_i)$, $\theta'_i = (v'_i, b_i)$, and $\theta_{-i} = (v_{-i}, b_{-i})$ we have that $v_i \leq v'_i$ implies $\sum_{j \in M} x_{i,j}(\theta_i, \theta_{-i}) \alpha_j \leq \sum_{j \in M} x_{i,j}(\theta'_i, \theta_{-i}) \alpha_j$. Mechanism $M = (x, p)$ for single-dimensional valuations and indivisible items that respects the publicly known budgets satisfies *payment identity* (PI) if for all $i \in N$ and $\theta = (v, b)$ with $c_{\gamma_t} \leq v_i \leq c_{\gamma_{t+1}}$ we have $p_i(\theta) = p_i((0, b_i), \theta_{-i}) + \sum_{s=1}^t (\gamma_s - \gamma_{s-1}) c_{\gamma_s}(b_i, \theta_{-i})$, where $\gamma_0 < \gamma_1 < \dots$ are the values $\sum_{j \in M} x_{i,j} \alpha_j$ can take and $c_{\gamma_s}(b_i, \theta_{-i})$ for $1 \leq s \leq t$ are the corresponding critical valuations. While VM ensures that stating a higher valuation can only lead to a better allocation, PI gives a formula for the payment in terms of the possible allocations and the critical valuations. In the proof of our impossibility result we will use the fact

⁴For PO we only need that the outcome respects the *reported* budget limits. Hence our characterization also applies in *private* budget settings.

that the payments for worse allocations provide a lower bound on the payments for better allocations.

Proposition 5. *Mechanism $M = (x, p)$ for single-dimensional valuations and indivisible items that respects the publicly known budgets is IC if and only if it satisfies VM and PI.*

Deterministic Mechanisms for Indivisible Items. The proof of our impossibility result uses the characterizations of PO outcomes and mechanisms that are IC with public budgets as follows: (a) PO is characterized by NT and NT induces a lower bound on the agents' payments for a *specific* assignment, namely for the case that agent 1 only gets item m . (b) IC, in turn, is characterized by VM and PI. Now VM and PI can be used to extend the lower bound on the payments for the *specific* assignment to *all* possible assignments. (c) Finally, IR implies upper bounds on the payments that, with a suitable choice of valuations, conflict with the lower bounds on the payments induced by NT, VM, and PI.

Theorem 3. *For single-dimensional valuations, indivisible items, and public budgets there can be no deterministic mechanism $M = (x, p)$ that satisfies IR, PO, and IC.*

Proof. For a contradiction suppose that there is a mechanism $M = (x, p)$ that is IR, PO, and IC for all n and all m . Consider a setting with $n = 2$ agents and $m = 2$ items in which $v_1 > v_2 > 0$ and $b_1 > \alpha_1 v_2$.

Observe that if agent 1's valuation was $v'_1 = 0$ and he reported his valuation truthfully, then since M satisfies IR his utility would be $u_1((0, b_1), \theta_{-1}, (0, b_1)) = -p_1((0, b_1), \theta_{-1}) \geq 0$. This shows that $p_1((0, b_1), \theta_{-1}) \leq 0$.

By PO, which by Proposition 4 is characterized by NT, agent 1 with valuation $v_1 > v_2$ and budget $b_1 > \alpha_1 v_2$ must win at least one item because otherwise he could buy any item from agent 2 and compensate him for his loss.

PO, respectively NT, also implies that agent 1's payment for item 2 must be strictly larger than $b_1 - (\alpha_1 - \alpha_2)v_2$ because otherwise he could trade item 2 against item 1 and compensate agent 2 for his loss.

By IC, which by Proposition 5 is characterized by VM and PI, agent 1's payment for item 2 is given by $p_1(\{2\}) = p_1((0, b_1), \theta_{-1}) + \alpha_2 c_{\alpha_2}(b_1, \theta_{-1})$, where c_{α_2} is the critical valuation for winning item 2. Together with $p_1(\{2\}) > b_1 - (\alpha_1 - \alpha_2)v_2$ this shows that $c_{\alpha_2}(b_1, \theta_{-1}) > (1/\alpha_2)[b_1 - (\alpha_1 - \alpha_2)v_2 - p_1((0, b_1), \theta_{-1})]$.

IC, respectively VM and PI, also imply that agent 1's payment for any non-empty set of items S in terms of the fractions $\gamma_t = \sum_{j \in S} \alpha_j > \dots > \gamma_1 = \alpha_2 > \gamma_0 = 0$ and corresponding critical valuations $c_{\gamma_t}(b_1, \theta_{-1}) \geq \dots \geq c_{\gamma_1}(b_1, \theta_{-1}) = c_{\alpha_2}(b_1, \theta_{-1})$ is $p_1(S) = p_1((0, b_1), \theta_{-1}) + \sum_{s=1}^t (\gamma_s - \gamma_{s-1}) c_{\gamma_s}(b_1, \theta_{-1})$. Because $c_{\gamma_s}(b_1, \theta_{-1}) \geq c_{\alpha_2}(b_1, \theta_{-1})$ for all s and $\sum_{s=1}^t (\gamma_s - \gamma_{s-1}) = \sum_{j \in S} \alpha_j$ we obtain $p_1(S) \geq p_1((0, b_1), \theta_{-1}) + (\sum_{j \in S} \alpha_j) c_{\alpha_2}(b_1, \theta_{-1})$.

Combining this lower bound on $p_1(S)$ with the lower bound on $c_{\alpha_2}(b_1, \theta_{-1})$ shows that $p_1(S) > (\sum_{j \in S} \alpha_j / \alpha_2) [b_1 - (\alpha_1 - \alpha_2)v_2]$.

For v_1 such that $(1/\alpha_2)[b_1 - (\alpha_1 - \alpha_2)v_2] > v_1 > v_2$ we know that agent 1 must win some item, but for any non-empty set of items S the lower bound on agent 1's payment for S contradicts IR. \square

Randomized Mechanisms for Indivisible and Divisible Items. Interestingly, the impossibility result for deterministic mechanisms for indivisible items can be avoided by a randomized mechanism: (a) Apply the randomized mechanism for a single *indivisible* item of [4] to a single indivisible item for which agent $i \in N$ has valuation $\tilde{v}_i = \sum_{j \in M} \alpha_j v_i$. (b) Map the single-item outcome (\tilde{x}, \tilde{p}) into an outcome (x, p) for the multi-item setting by setting $x_{i,j} = 1$ for all $j \in M$ if and only if $\tilde{x}_i = 1$ and setting $p_i = \tilde{p}_i$ for all $i \in N$.

A similar idea works for divisible items. The only difference is that we use the mechanisms of [4] for a single *divisible* item, and map the single-item outcome (\tilde{x}, \tilde{p}) into a multi-item outcome by setting $x_{i,j} = \tilde{x}_i$ for all $i \in N$ and all $j \in M$ and setting $p_i = \tilde{p}_i$ for all $i \in N$.

The main difficulty in proving that the resulting mechanisms inherit the properties of the mechanisms in [4] is to show that the resulting mechanisms satisfy PO (ex post). For this we argue that a certain structural property of the single-item outcomes is preserved by the mapping to the multi-item setting and remains to be sufficient for PO (ex post).

Proposition 6. *Let (\tilde{x}, \tilde{p}) be the outcome of our mechanism and let (x, p) be the outcome of the respective mechanism of [4], then $u_i(\tilde{x}_i, \tilde{p}_i) = u_i(x_i, p_i)$ for all $i \in N$ resp. $E[u_i(\tilde{x}_i, \tilde{p}_i)] = E[u_i(x_i, p_i)]$ for all $i \in N$.*

Theorem 4. *For single-dimensional valuations, divisible or indivisible items, and private budgets there is a randomized mechanism that satisfies IR ex interim, NPT ex post, PO ex post, and IC ex interim. Additionally, for single-dimensional valuations and divisible items there is a deterministic mechanism that satisfies IR, NPT, PO, and IC for public budgets.*

Proof. IR (ex interim) and IC (ex interim) follow from Proposition 6 and the fact that the mechanisms of [4] are IR (ex interim) and IC (ex interim). NPT (ex post) follows from the fact that the payments in our mechanisms and the mechanisms of [4] are the same, and the mechanisms in [4] satisfy NPT (ex post). For PO (ex post) we argue that the structural property of the outcomes of the mechanisms in [4] that (a) $\sum_{i \in N} \tilde{x}_{i,j} = 1$ for all $j \in M$ and (b) $\sum_{j \in M} \tilde{x}_{i,j} > 0$ and $\tilde{v}_{i'} > \tilde{v}_i$ imply $\tilde{p}_{i'} = b_{i'}$ is preserved by the mapping to the multi-item setting and remains to be sufficient for PO (ex post).

We begin by showing that the structural property is preserved by the mapping. For this observe that $\sum_{i \in N} \tilde{x}_{i,j} = 1$ for all $j \in M$ implies that $\sum_{i \in N} x_{i,j} = 1$ for all $j \in M$ and that $\sum_{j \in M} \tilde{x}_{i,j} > 0$ and $\tilde{v}_{i'} > \tilde{v}_i$ imply $\tilde{p}_{i'} = b_{i'}$ implies that $\sum_{j \in M} x_{i,j} > 0$ and $v_{i'} > v_i$ imply $p_{i'} = b_{i'}$.

Next we show that the structural property remains to be sufficient for PO (ex post). For this assume by contradiction that the outcome (x, p) is *not* PO (ex post). Then, by Proposition 4, there exists an x' such that $\sum_{i \in N} \delta_i v_i > 0$ and $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) +$

$\sum_{i \in L} \delta_i v_i \geq 0$, where $\delta_i = \sum_{j \in M} (x'_{i,j} - x_{i,j}) \alpha_j$, $W = \{i \in N \mid \delta_i > 0\}$, and $L = \{i \in N \mid \delta_i \leq 0\}$.

Because (x, p) satisfies condition (a), i.e., $\sum_{i \in N} x_{i,j} = 1$ for all $j \in M$, and x' is a valid assignment, i.e., $\sum_{i \in N} x'_{i,j} \leq 1$ for all $j \in M$, we have $\sum_{i \in N} \delta_i = \sum_{j \in M} \sum_{i \in N} (x'_{i,j} - x_{i,j}) \alpha_j \leq 0$. Because $\sum_{i \in N} \delta_i v_i > 0$ we have $\sum_{i \in W} \delta_i v_i \geq \sum_{i \in N} \delta_i v_i > 0$ and, thus, $\sum_{i \in W} \delta_i > 0$. We conclude that $\sum_{i \in L} \delta_i = \sum_{i \in N} \delta_i - \sum_{i \in W} \delta_i < 0$ and, thus, $\sum_{i \in L} \delta_i v_i < 0$.

Because (x, p) satisfies condition (b), i.e., $\sum_{j \in M} x_{i,j} > 0$ and $v_{i'} > v_i$ imply $p_{i'} = b_{i'}$, there exists a t with $1 \leq t \leq n$ such that (1) $\sum_{j \in M} x_{i,j} \geq 0$ and $p_i = b_i$ for $1 \leq i \leq t$, (2) $\sum_{j \in M} x_{i,j} \geq 0$ and $p_i \leq b_i$ for $i = t+1$, and (3) $\sum_{j \in M} x_{i,j} = 0$ and $p_i \leq b_i$ for $t+2 \leq i \leq n$.

Case 1: $t = n$. Then $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) = 0$ and, thus, $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i < 0$.

Case 2: $t < n$ and $W \cap \{1, \dots, t\} = \emptyset$. Then $\sum_{i \in W} \delta_i v_i \leq \sum_{i \in W} \delta_i v_{t+1}$ and $\sum_{i \in L} \delta_i v_i \leq \sum_{i \in L} \delta_i v_{t+1}$ and, thus, $\sum_{i \in N} \delta_i v_i = \sum_{i \in W} \delta_i v_i + \sum_{i \in L} \delta_i v_i \leq \sum_{i \in N} \delta_i v_{t+1} \leq 0$.

Case 3: $t < n$ and $W \cap \{1, \dots, t\} \neq \emptyset$. Then $\sum_{i \in W} \min(p_i - b_i, \delta_i v_i) \leq \sum_{i \in W \setminus \{1, \dots, t\}} \delta_i v_{t+1}$ and $\sum_{i \in L} \delta_i v_i \leq \sum_{i \in L} \delta_i v_{t+1}$ and, thus, $\sum_{i \in W} \min(p_i - b_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \leq (\sum_{i \in N} \delta_i - \sum_{i \in W \cap \{1, \dots, t\}} \delta_i) v_{t+1} < 0$. \square

5 Conclusion and Future Work

In this paper we analyzed IR, PO, and IC mechanisms for settings with heterogeneous items. Our main accomplishments are: (a) An impossibility result for *randomized* mechanisms and *public* budgets for additive valuations. (b) *Randomized* mechanisms that achieve these properties for *private* budgets and a restricted class of additive valuations. We are able to circumvent the impossibility result in the restricted setting because our argument for the impossibility result is based on the ability of an agent to overstate his valuation for one and understate his valuation for another item, which is not possible in the restricted setting. A promising direction for future work is to identify other valuations for which this is the case.

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A Proof of Proposition 1

Fix $i \in N$ and $\theta_{-i} = (v_{-i}, b_{-i})$. By IC agent i does not benefit from reporting $\theta'_i = (v'_i, b_i)$ when his true type is $\theta_i = (v_i, b_i)$, nor does he benefit from reporting $\theta_i = (v_i, b_i)$ when his true type is $\theta'_i = (v'_i, b_i)$. Thus,

$$\begin{aligned} v_i(x(\theta_i, \theta_{-i})) - p_i(\theta_i, \theta_{-i}) &\geq v_i(x(\theta'_i, \theta_{-i})) - p_i(\theta'_i, \theta_{-i}) \\ v'_i(x(\theta'_i, \theta_{-i})) - p_i(\theta'_i, \theta_{-i}) &\geq v'_i(x(\theta_i, \theta_{-i})) - p_i(\theta_i, \theta_{-i}) \end{aligned}$$

By combining these inequalities we get

$$v'_i(x_i(\theta'_i, \theta_{-i})) - v'_i(x_i(\theta_i, \theta_{-i})) \geq v_i(x_i(\theta'_i, \theta_{-i})) - v_i(x_i(\theta_i, \theta_{-i})).$$

B Analysis of Cases 1 and 2 in Section 3

We start the analysis with an auxiliary lemma that shows that if at least one agent $i \in N$ has a positive valuation for some item $j \in M$ then this item j must be assigned completely in every outcome that is IR and PO.

Lemma 3. *If the valuation of at least one agent for an item $j \in M$ is positive, then an IR and PO outcome assigns all of item j , i.e., $\sum_{i=1}^n x_{i,j} = 1$.*

Proof. Let us assume by contradiction that we have an outcome (x, p) in which not all of the fractions of item j are assigned to the agents. Then the utility of the agents who have a positive valuation strictly increases when they get the unsold fractions of item j at price 0, while the utility of the other agents and that of the auctioneer remain unchanged. Contradiction to PO! \square

Case 1 is easy: Agent 2 is not budget restricted and has the highest valuations for both items; so he will get both items. Thus in this case the utility for agent 1 is zero.

Lemma 4. *Given $b_2 > v_{2,1} + v_{2,2}$, $v_{2,1} > v_{1,1}$ and $v_{2,2} > v_{1,2}$, then $x_{1,1} = 0$, $x_{1,2} = 0$, $x_{2,1} = 1$, $x_{2,2} = 1$, and $u_1 = 0$ in every IR and PO outcome selected by an IC mechanism.*

Proof. We divide the proof into the following parts: in (a) we show that $x_{1,1} = 0$, $x_{1,2} = 0$, $x_{2,1} = 1$, and $x_{2,2} = 1$, and in (b) we show that $u_1 = 0$.

- (a) Let us assume by contradiction that we have an IR and PO outcome where $x_{1,1} > 0$ or $x_{1,2} > 0$. IR requires that $p_2 \leq x_{2,1}v_{2,1} + x_{2,2}v_{2,2}$. Hence, agent 2 can buy the fractions $x_{1,1}$ of item 1 and $x_{1,2}$ of item 2 for a payment p with $x_{1,1}v_{2,1} + x_{1,2}v_{2,2} > p \geq x_{1,1}v_{1,1} + x_{1,2}v_{1,2}$ from agent 1. Because of $v_{2,1} > v_{1,1}$ and $v_{2,2} > v_{1,2}$ such a payment exists and agent 2 has enough money, since $b_2 > v_{2,1} + v_{2,2}$ implies

$$b_2 > v_{2,1} + v_{2,2} = (x_{1,1} + x_{2,1})v_{2,1} + (x_{1,2} + x_{2,2})v_{2,2} > p_2 + p. \quad (1)$$

The utility of agent 2 would increase and the utilities of agent 1 and the auctioneer would not decrease. Contradiction to PO!

- (b) We have already shown before that agent 1 gets no fraction of the items, and therefore, IR implies that his payments cannot be positive.

Let us consider the subcase where $v_{1,1} = v_{1,2} = 0$ and agent 1 reports truthfully. The valuations of agent 2 are positive. Because of IR the payment of agent 2 cannot exceed his reported valuation, but (a) holds when his reported valuations are positive. Therefore, agent 2 would have an incentive to understate his valuation when his payment would be positive. Hence, IR of the auctioneer implies that the payment of both agents is equal to 0. This means, that the utility of agent 1 is 0 in this case.

If there would exist any other reported valuation of agent 1, where he gets no items, but where his payments are negative, then he would have an incentive to lie, when his valuations are equal to 0. This would contradict IC!

\square

In Case 2, agent 1 has the higher valuation for item 1, while agent 2 has the higher valuation for item 2. Thus, agent 1 gets item 1 and agent 2 gets item 2. Since the only difference to Case 1 is that in Case 2 $v_{1,1} > v_{2,1}$ while in Case 1 $v_{1,1} < v_{2,1}$, the critical value whether agent 2 gets item 1 or not is $v_{2,1}$, and thus in every IC mechanism, agent 1 has to pay $v_{2,1}$ and his utility is $v_{1,1} - v_{2,1}$.

Lemma 5. *Given $b_2 > v_{2,1} + v_{2,2}$, $v_{1,1} > v_{2,1}$, $v_{2,2} > v_{1,2}$, and $b_1 > v_{1,1}$, then $x_{1,1} = 1$, $x_{1,2} = 0$, $x_{2,1} = 0$, $x_{2,2} = 1$, and $u_1 = v_{1,1} - v_{2,1}$ in every IR and PO outcome selected by an IC mechanism.*

Proof. We divide the proof into the following parts: in (a) we show that $x_{1,1} = 1$, $x_{1,2} = 0$, $x_{2,1} = 0$, and $x_{2,2} = 1$, and in (b) we show that $u_1 = v_{1,1} - v_{2,1}$.

- (a) Let us assume by contradiction that $x_{1,2} > 0$. Then, agent 2 can buy these fractions of item 2 for a payment p with $x_{1,2}v_{2,2} > p \geq x_{1,2}v_{1,2}$, which exists because of $v_{2,2} > v_{1,2}$. IR and $b_2 > v_{2,1} + v_{2,2}$ ensure that agent 2 has enough budget, since $b_2 > v_{2,1} + v_{2,2} = (x_{1,1} + x_{2,1})v_{2,1} + (x_{1,2} + x_{2,2})v_{2,2} \geq p_2 + x_{1,1}v_{2,1} + x_{1,2}v_{2,2} > p_2 + p$. The utility of the agent 2 would increase, while the utilities of agent 1 and the auctioneer would not decrease. Contradiction to PO!

Otherwise, let us assume that $x_{1,1} < 1$ and $x_{1,2} = 0$. Then, agent 1 can buy the other fractions of item 1 for a payment p with $x_{2,1}v_{1,1} > p \geq x_{2,1}v_{2,1}$, which exists because of $v_{1,1} > v_{2,1}$. IR and $b_1 > v_{1,1}$ ensure that agent 1 has enough budget, since $b_1 > v_{1,1} = (x_{1,1} + x_{2,1})v_{1,1} \geq p_1 + x_{2,1}v_{1,1} > p_1 + p$. The utility of agent 1 would increase, while the utilities of agent 2 and the auctioneer would not decrease. Contradiction to PO!

- (b) We show first that $p_1 \leq v_{2,1}$. Since $x_{1,1} = 1$ and $x_{1,2} = 0$, IR requires that $p_1 \leq v_{1,1}$. If $p_1 > v_{2,1}$, then agent 1 has an incentive to lie. If he states that his valuation for item 1 is $v'_{1,1}$ with $p_1 > v'_{1,1} > v_{2,1}$, then the allocation of the items does not change, but he pays less because of IR. Contradiction to IC!

Now, we show that $p_1 \geq v_{2,1}$. Let us therefore assume by contradiction that $p_1 < v_{2,1}$. If we have $v'_{1,1}$ with $p_1 < v'_{1,1} < v_{2,1}$ instead of $v_{1,1}$, and all the other valuations are left unchanged, then Lemma 4 implies that $u'_1 = 0$. Hence, in this case agent 1 can increase his utility when he lies and states that his valuation is $v_{1,1}$, because his utility would be $v'_{1,1} - p_1 > 0$. Contradiction to IC!

Since agent 1 gets all fractions of item 1, no fraction of item 2, and has to pay $v_{2,1}$, his utility is $v_{1,1} - v_{2,1}$.

□

C Proof of Lemma 1

We divide the proof into the following parts: in (a) we show that $x_{1,1} = 1$ and $x_{1,2} = 1$ if $p_1 < b_1$, in (b) we show that $x_{1,2} > (1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}}$ if $p_1 = b_1$, and in (c) we show that $x_{1,1} = 1$ and $x_{1,2} > 0$ if $p_1 = b_1$.

- (a) Let us assume by contradiction that $p_1 < b_1$ and $x_{1,j} < 1$ for an item $j \in \{1, 2\}$. Agent 1 can increase his utility by buying $\min\{\frac{b_1 - p_1}{p}, x_{2,j}\}$ fractions of item j for a unit price p with $v_{1,j} > p \geq v_{2,j}$ from agent 2. Such a price exists, because of $v_{1,1} > v_{2,1}$ and $v_{1,2} > v_{2,2}$. Agent 1 has enough money for the trade, since $p_1 + p \min\{\frac{b_1 - p_1}{p}, x_{2,j}\} = \min\{b_1, p_1 + px_{2,j}\} \leq b_1$. The utility of agent 1 would increase, and the utilities of agent 2 and the auctioneer would not decrease. Contradiction to PO!
- (b) IR requires $b_1 = p_1 \leq v_{1,1}x_{1,1} + v_{1,2}x_{1,2}$, and therefore, $x_{1,2} \geq \frac{b_1 - v_{1,1}x_{1,1}}{v_{1,2}}$. If $x_{1,1} = 1$, then $b_1 > v_{1,1}$ implies that $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}} = 0 < \frac{b_1 - v_{1,1}}{v_{1,2}} = \frac{b_1 - v_{1,1}x_{1,1}}{v_{1,2}}$. Otherwise, if $x_{1,1} = 0$, then $b_1 > v_{1,1}$ and $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$ imply that $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}} = \frac{v_{2,1}}{v_{2,2}} < \frac{b_1}{v_{1,2}} = \frac{b_1 - v_{1,1}x_{1,1}}{v_{1,2}}$, and hence, $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}} < \frac{b_1 - v_{1,1}x_{1,1}}{v_{1,2}}$ for all $x_{1,1} \in [0, 1]$. Therefore, we have that $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}} < x_{1,2}$ for all possible values of $x_{1,1}$.
- (c) We split the proof into two parts. We assume by contradiction that either $p_1 = b_1$, $x_{1,1} \leq 1$ and $x_{1,2} = 0$, or that $p_1 = b_1$, $x_{1,1} < 1$ and $x_{1,2} > 0$.

Let us assume that $p_1 = b_1$, $x_{1,1} \leq 1$ and $x_{1,2} = 0$. According to $b_1 > v_{1,1}$, the utility of agent 1 is negative. Contradiction to IR!

We will now investigate the other case and assume that $p_1 = b_1$, $x_{1,1} < 1$ and $x_{1,2} > 0$. Agent 2 has the same valuation for $x_{1,2} = 1 - x_{1,1}$ fractions of item 1 and $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}}$ fractions of item 2. The valuation of agent 1 for $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}}$ fractions of item 2 is identical to the valuation for $(1 - x_{1,1})\frac{v_{2,1}v_{1,2}}{v_{2,2}v_{1,1}}$ fractions of item 1. We know that $v_{2,1}v_{1,2} < v_{2,2}v_{1,1}$. That is, that the utility of agent 1 is increased and the utilities of agent 2 and the auctioneer are not decreased, when agent 1 trades $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}}$ fractions of item 2 against $x_{2,1} = 1 - x_{1,1}$ fractions of item 1. Fact (b) implies that agent 1 actually has the required $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}}$ fractions of item 2. Contradiction to PO!

D Proof of Lemma 2

We divide the proof into the following parts: in (a) we show that $p_1 = b_1$ and $x_{1,2} < 1$, in (b) we show that $\frac{b_1 - v_{2,1}}{v_{2,2}} \geq x_{1,2} \geq \frac{b_1 - v_{2,1}}{v_{1,2}}$, and in (c) we show that $x_{1,2} = \frac{b_1 - v_{2,1}}{v_{2,2}}$.

- (a) Lemma 1 implies that the utility of agent 1 is $v_{1,1} + x_{1,2}v_{1,2} - p_1$. We know that $v_{2,1} + v_{2,2} > b_1$. Hence, we can select a sufficiently small $\epsilon > 0$ such that $v_{2,1} + v_{2,2} - \epsilon >$

b_1 . Because of $v_{1,1} > v_{2,1}$ and $b_1 > v_{1,1}$, we know that $v_{2,2} - \epsilon > 0$. Let us consider the case where we have $v'_{1,2}$ with $v_{2,2} > v'_{1,2} > v_{2,2} - \epsilon$ instead of $v_{1,2}$ and all other valuation are left unchanged. In this case, the utility of agent 1 is $v_{1,1} - v_{2,1}$, because of Lemma 5 and since $v_{2,2} > v'_{1,2}$ holds. Therefore, IC implies that

$$v_{1,1} - v_{2,1} \geq v_{1,1} + x_{1,2}v'_{1,2} - p_1. \quad (2)$$

Let us assume by contradiction that $x_{1,2} = 1$, then equation (2) implies

$$p_1 \geq v_{2,1} + v'_{1,2} > v_{2,1} + v_{2,2} - \epsilon > b_1, \quad (3)$$

which contradicts the budget constraint. Therefore, $x_{1,2} < 1$, and hence, Lemma 1 implies that $p_1 = b_1$.

- (b) Lemma 1 and (a) show that the utility of agent 1 is $v_{1,1} + x_{1,2}v_{1,2} - b_1$. We select a sufficiently small $\epsilon > 0$, such that $v_{2,1} + v_{2,2} - \epsilon > b_1$ and consider the case where $v'_{1,2} = v_{2,2} - \epsilon$ and all other valuations are unchanged. Lemma 5 implies that the utility of agent 1 is $v_{1,1} - v_{2,1}$ in this case. Hence, IC implies that

$$v_{1,1} - v_{2,1} \geq v_{1,1} + x_{1,2}v'_{1,2} - b_1 \quad (4)$$

and

$$v_{1,1} + x_{1,2}v_{1,2} - b_1 \geq v_{1,1} - v_{2,1}. \quad (5)$$

Inequality (4) implies that $\frac{b_1 - v_{2,1}}{v_{2,2} - \epsilon} = \frac{b_1 - v_{2,1}}{v'_{1,2}} \geq x_{1,2}$. Since this inequality has to hold for all sufficiently small $\epsilon > 0$, we know that $\frac{b_1 - v_{2,1}}{v_{2,2}} \geq x_{1,2}$. Inequality (5) implies that $\frac{b_1 - v_{2,1}}{v_{1,2}} \leq x_{1,2}$.

- (c) Let us assume by contradiction that the inequality $\frac{b_1 - v_{2,1}}{v_{2,2}} \geq x_{1,2}$ implied by (b) is strict, and $\gamma > 0$ is defined such that $\frac{b_1 - v_{2,1}}{v_{2,2}} = x_{1,2} + \gamma$. We select arbitrary $\epsilon > 0$ and δ with $v_{2,2} \left(\frac{b_1 - v_{2,1}}{b_1 - v_{2,1} - \gamma v_{2,2}} - 1 \right) > \delta > 0$ which fulfill $v_{1,2} - \epsilon - \delta = v_{2,2}$. Such variables ϵ and δ exist because of $v_{1,2} > v_{2,2}$, and since $v_{1,1} > v_{2,1}$, $b_1 > v_{1,1}$ and $\gamma > 0$ imply that $\frac{b_1 - v_{2,1}}{b_1 - v_{2,1} - \gamma v_{2,2}} > 1$. We consider the alternative case where $v'_{1,2} = v_{1,2} - \epsilon$ and all other valuations are unchanged. In this case, (b) implies that $\frac{b_1 - v_{2,1}}{v'_{1,2}} \leq x'_{1,2}$, and hence, $\frac{b_1 - v_{2,1}}{v_{2,2} + \delta} \leq x'_{1,2}$. Furthermore, Lemma 1 and (a) imply that $p_1 = b_1$ and $x_{1,1} = 1$ in both cases. Now, IC requires that

$$v_{1,1} + x_{1,2}v_{1,2} - b_1 \geq v_{1,1} + x'_{1,2}v_{1,2} - b_1, \quad (6)$$

respectively $x_{1,2} \geq x'_{1,2}$, and therefore, $\frac{b_1 - v_{2,1}}{v_{2,2}} - \gamma \geq \frac{b_1 - v_{2,1}}{v_{2,2} + \delta}$. But this inequality can be transformed to $\delta \geq v_{2,2} \left(\frac{b_1 - v_{2,1}}{b_1 - v_{2,1} - \gamma v_{2,2}} - 1 \right)$. Contradiction!

E Adaptive Clinching Auction for a Single Divisible Item

We investigate the adaptive clinching auction for a single divisible item that is described as a “continuous time process” in [4] in order to construct an explicit algorithm. In Step (II) of the differential process described in [4] the item is overdemanded and no bidder exits the auction because his valuation is identical to the current price. This is the case which has to be analyzed. We consider a time span $[t_1, t_2]$. A is the set of active bidders at time t_1 and C is the set of clinching bidders at time t_1 . For all $t \in [t_1, t_2]$ we have $p(t) = p(t_1) + (t - t_1)$. We assume that t_2 is selected such that $v_i > p(t)$ for all $i \in A$ and $t \in [t_1, t_2]$. Therefore, the set of active bidders at time $t \in [t_1, t_2]$ is equal to A and the set of exiting bidders at time $t \in [t_1, t_2]$ is empty. We assume further that t_2 is selected such that no bidder starts clinching during (t_1, t_2) , and that the demand $D(t) = \sum_{i \in A} \frac{b_i(t)}{p(t)}$ is larger than the supply $S(t)$ for all $t \in [t_1, t_2]$. Hence, at every time t in (t_1, t_2) Step (II) of the process is selected and the set of clinching bidders C does not change.

Consider time t in (t_1, t_2) . By the definition of the clinching bidders the supply is given by $S(t) = \sum_{j \in A \setminus \{i\}} \frac{b_j(t)}{p(t)}$ for all $t \in (t_1, t_2)$ and every clinching bidder $i \in C$. Since every clinching bidder $i \in C$ gets the same fraction allocated during (t_1, t) we have

$$x_i(t) - x_i(t_1) = \frac{S(t_1) - S(t)}{|C|} = \frac{1}{|C|} \sum_{j \in A \setminus \{i\}} \left(\frac{b_j(t_1)}{p(t_1)} - \frac{b_j(t)}{p(t)} \right).$$

Let us now differentiate this equation with respect to t . We get

$$x'_i(t) = \frac{1}{|C|} \sum_{j \in A \setminus \{i\}} \left(-\frac{b'_j(t)p(t) - b_j(t)p'(t)}{p(t)^2} \right) = \frac{1}{|C|} \sum_{j \in A \setminus \{i\}} \left(\frac{b_j(t)}{p(t)^2} - \frac{b'_j(t)}{p(t)} \right).$$

Bidder $i \in C$ pays $p(t)$ for the fractions that he is clinching at time t . Hence $b'_i(t) = -x'_i(t)p(t)$. This, the previous equality, $b'_j(t) = 0$ for $j \in A \setminus C$, and $b'_j(t) = b'_i(t)$ and $b_j(t) = b_i(t)$ for $j \in C \setminus \{i\}$ implies

$$b'_i(t) = - \sum_{j \in A \setminus \{i\}} \frac{b_j(t)}{p(t)} = \frac{-\sum_{j \in A \setminus C} b_j(t_1) - (|C| - 1)b_i(t)}{p(t_1) - t_1 + t}.$$

For the case that $|C| > 1$ we can solve this differential equation and obtain

$$b_i(t) = \frac{1}{|C| - 1} \left(\left(\frac{p(t_1)}{p(t)} \right)^{|C|-1} \sum_{j \in A \setminus \{i\}} b_j(t_1) - \sum_{j \in A \setminus C} b_j(t_1) \right).$$

Since $b_j(t) = b_j(t_1)$ for all $j \in A \setminus C$ and $b_j(t) = b_i(t)$ for all $j \in C$ it follows that

$$|C|^{-1} \sqrt{\frac{\sum_{j \in A \setminus \{i\}} b_j(t_1)}{\sum_{j \in A \setminus \{i\}} b_j(t)}} = \frac{p(t)}{p(t_1)}. \quad (7)$$

For the case that $|C| = 1$ we have

$$b'_i(t) = -\frac{\sum_{j \in A \setminus \{i\}} b_j(t_1)}{p(t_1) - t_1 + t}$$

and obtain

$$\exp\left(\frac{b_i(t_1) - b_i(t)}{\sum_{j \in A \setminus \{i\}} b_j(t_1)}\right) = \frac{p(t)}{p(t_1)}. \quad (8)$$

Equations (7) and (8) allow us to compute the prices where a new bidder would start clinching.

The construction of the algorithm follows the differential process described in [4]. Lines 28-35 of Algorithm 1 correspond to Step (I) in [4]; lines 13-23 of Algorithm 1 correspond to Step (III) in [4]; and Algorithm 2, which is called on line 10 of Algorithm 1 corresponds to Step (II) in [4]. Line 16, 20, and 22 of Algorithm 2 follow from equation (7) and (8). The variables that are used in Algorithm 1 and Algorithm 2 are described in Table 3.

For the running time observe that each time one of the two while-loops gets executed either an active bidder who was not clinching becomes a clinching bidder or an active bidder becomes an exiting bidder. Since an exiting bidder cannot become active again and an active clinching bidder cannot become an active non-clinching bidder again it follows that the algorithm runs in time polynomial in the bidders.

Table 3: Description of the Variables in Algorithm 1

<i>Variable</i>	<i>Data Type</i>	<i>Constraint</i>	<i>Description</i>
n	integer (constant)	$n > 1$	number of bidders
b	real vector (length n)	$b_i > 0 \forall i \in \{1, \dots, n\}$	budgets
v	real vector (length n)	$v_i > 0 \forall i \in \{1, \dots, n\}$	valuations
A	set of integers	$A \subseteq \{1, \dots, n\}$	set of active bidders
E	set of integers	$E \subseteq \{1, \dots, n\}$	set of exiting bidders
C	set of integers	$C \subseteq \{1, \dots, n\}$	set of clinching bidders
p	real	$p \geq 0$	price
x	real vector (length n)	$x_i \geq 0 \forall i \in \{1, \dots, n\}$	allocated amount
S	real	$S \geq 0$	supply
D	real	$D \geq 0$	aggregated demand

Algorithm 1 Adaptive Clinching Auction for a Single Divisible Good.

```

1: procedure CLINCHING( $n, b, v$ )
2:    $\backslash\backslash$  initialize variables
3:    $(p, S, D) \leftarrow (0, 1, \infty)$ 
4:    $(A, E, C) \leftarrow (\{1, \dots, n\}, \emptyset, \emptyset)$ 
5:    $x_i \leftarrow 0 \forall i \in A$ 
6:   while  $D > S$  do

```

Algorithm 1 Adaptive Clinching Auction for a Single Divisible Good (Continued).

```
7:      \\ item is overdemanded
8:      if  $E = \emptyset$  then
9:          \\ there are no exiting bidders
10:          $(A, E, C, p, S, D, x, b) \leftarrow \text{CONTINUOUSCLINCHING}(A, C, p, S, D, x, b, v)$ 
11:     else
12:         \\ there are exiting bidders
13:          $m \leftarrow \sum_{i \in E} b_i$ 
14:          $E \leftarrow \emptyset$ 
15:         while  $m > 0$  do
16:             \\ compute amount that the clinching bidders can clinch
17:             \\ before a new bidder starts clinching
18:              $c \leftarrow \min\{|C|(\min_{i \in C} b_i - \max_{i \in A \setminus C} b_i), m\}$ 
19:              $m \leftarrow m - c$ 
20:              $(x_i, b_i) \leftarrow (x_i + \frac{c}{|C|p}, b_i - \frac{c}{|C|}) \forall i \in C$ 
21:              $(S, D) \leftarrow (S - \frac{c}{p}, D - \frac{c}{p})$ 
22:              $C \leftarrow \{i \in A \mid D - \frac{b_i}{p} = S\}$ 
23:         end while
24:     end if
25: end while
26:     \\ item is not overdemanded
27:     \\ sell to active bidders the amount they can afford
28:      $S \leftarrow S - D$ 
29:      $(x_i, b_i) \leftarrow (x_i + \frac{b_i}{p}, 0) \forall i \in A$ 
30:     \\ sell left fractions to exiting bidders
31:     for  $i \in E$  do
32:          $m \leftarrow \min\{\frac{b_i}{p}, S\}$ 
33:          $(x_i, b_i) \leftarrow (x_i + m, b_i - mp)$ 
34:          $S \leftarrow S - m$ 
35:     end for
36:     return  $(x, b)$ 
37: end procedure
```

Algorithm 2 Continuous Clinching.

```
1: procedure CONTINUOUSCLINCHING( $A, C, p, S, D, x, b, v$ )
2:     if  $C = \emptyset$  then
3:         \\ compute highest price  $p$  where no bidder clinched
4:         \\ or exited the auction before
5:          $p^* \leftarrow \frac{\sum_{i \in A} b_i - \max_{i \in A} b_i}{S}$ 
6:          $p \leftarrow \min\{p^*, \min_{i \in A} v_i\}$ 
7:         \\ update variables
8:          $(A, E) \leftarrow (\{i \in A \mid v_i > p\}, \{i \in A \mid v_i = p\})$ 
```

Algorithm 2 Continuous Clinching (Continued).

```
9:    $D \leftarrow \frac{\sum_{i \in A} b_i}{p}$ 
10:   $C \leftarrow \{i \in A \mid D - \frac{b_i}{p} = S\}$ 
11:  else
12:    \\\ compute next break point of the differential process
13:    \\\ and update variables
14:     $b^* \leftarrow \max_{j \in A \setminus C} b_j$ 
15:    \\\ price where a new bidder would start to clinch
16:     $p^* \leftarrow \begin{cases} p \exp\left(\frac{\max_{i \in A} b_i - b^*}{\sum_{i \in A \setminus C} b_i}\right), & \text{if } |C| = 1 \\ p \left( \frac{1}{|C| - 1} \sqrt{\frac{pS}{(|C| - 1)b^* + \sum_{i \in A \setminus C} b_i}} \right), & \text{if } |C| > 1 \end{cases}$ 
17:    \\\ price at the next break point
18:     $\tilde{p} \leftarrow \min\{p^*, \min_{i \in A} v_i\}$ 
19:    \\\ supply at the next break point
20:     $\tilde{S} \leftarrow \left(\frac{\tilde{p}}{p}\right)^{|C|} S$ 
21:    \\\ budget of the clinching bidder at next break point
22:     $\tilde{b} \leftarrow \begin{cases} \max_{i \in A} b_i - \log\left(\frac{\tilde{p}}{p}\right) \sum_{i \in A \setminus C} b_i, & \text{if } |C| = 1 \\ \frac{1}{|C| - 1} \left( S \frac{p^{|C|}}{\tilde{p}^{|C| - 1}} - \sum_{i \in A \setminus C} b_i \right), & \text{if } |C| > 1 \end{cases}$ 
23:    \\\ update variables
24:     $(x_i, b_i) \leftarrow (x_i + \frac{1}{|C|} (S - \tilde{S}), \tilde{b}) \forall i \in C$ 
25:     $E \leftarrow \{i \in A \mid v_i = p\}$ 
26:     $A \leftarrow \{i \in A \mid v_i > p\}$ 
27:     $C \leftarrow \arg \max_{i \in A} b_i$ 
28:     $(p, S, D) \leftarrow (\tilde{p}, \tilde{S}, \sum_{i \in A} \frac{b_i}{p})$ 
29:  end if
30:  return  $(A, E, C, p, S, D, x, b)$ 
31: end procedure
```

F Proof of Proposition 4

First we show that if (x, p) satisfies PO, then it satisfies NT. To this end we show that if (x, p) does *not* satisfy NT, then it is *not* PO.

Case 1: \neg NT because \neg (a)

There exists an item $j \in M$ such that $\sum_{i \in N} x_{i,j} < 1$. By assumption every agent $i \in N$ has $v_i > 0$ and, thus, $\alpha_j v_i > 0$. Consider the outcome (x', p') that results from assigning the unassigned fraction of item j to some agent $i' \in N$ at no additional cost. For this outcome we have $u'_i = u_i$ for all agents $i \in N \setminus \{i'\}$, $u'_{i'} > u_{i'}$ for agent i' , and $\sum_{i \in N} p'_i = \sum_{i \in N} p_i$. Hence (x, p) is *not* PO.

Case 2: \neg NT because \neg (b)

There exists an assignment x' such that $\sum_{i \in N} \delta_i v_i > 0$ and $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq 0$. Consider the outcome (x', p') for which $p'_i = p_i + \min(b_i - p_i, \delta_i v_i)$ for all

agents $i \in W$ and $p'_i = p_i + \delta_i v_i$ for all agents $i \in L$.

For all agents $i \in N$ we have $u'_i \geq u_i$ because

$$\begin{aligned}
u'_i &= \sum_{j \in M} x'_{i,j} \alpha_j v_i - p'_i \\
&= \sum_{j \in M} x_{i,j} \alpha_j v_i + \delta_i v_i - p_i - \min(b_i - p_i, \delta_i v_i) \\
&\geq u_i, && \text{for } i \in W, \text{ and} \quad (9) \\
u'_i &= \sum_{j \in M} x'_{i,j} \alpha_j v_i - p'_i \\
&= \sum_{j \in M} x_{i,j} \alpha_j v_i + \delta_i v_i - p_i - \delta_i v_i \\
&= u_i && \text{for } i \in L.
\end{aligned}$$

For the auctioneer we have $\sum_{i \in N} p'_i \geq \sum_{i \in N} p_i$ because

$$\begin{aligned}
\sum_{i \in N} p'_i - \sum_{i \in N} p_i &= \sum_{i \in W} p'_i + \sum_{i \in L} p'_i - \sum_{i \in N} p_i \\
&= \sum_{i \in W} (p_i + \min(b_i - p_i, \delta_i v_i)) + \sum_{i \in L} (p_i + \delta_i v_i) - \sum_{i \in N} p_i \\
&= \sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \\
&\geq 0. \tag{10}
\end{aligned}$$

If $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i > 0$, then inequality (10) is strict showing that $\sum_{i \in N} p'_i > \sum_{i \in N} p_i$. Otherwise, $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i = 0$, and since $\sum_{i \in N} \delta_i v_i > 0$ we must have $b_i - p_i < \delta_i v_i$ for at least one agent $i \in W$. For this agent i inequality (9) is strict showing that $u'_i > u_i$. Hence in both cases (x, p) is *not* PO.

Next we show that if (x, p) satisfies NT, then it is PO. To this end we show that if (x, p) is *not* PO, then it does *not* satisfy NT. If (x, p) is *not* PO, then there exists an outcome (x', p') such that $u'_i \geq u_i$ for all agents $i \in N$ and $\sum_i p'_i \geq \sum_i p_i$, with at least one of the inequalities strict.

If *not* all items are assigned completely in (x, p) , then we have \neg (a) and so (x, p) does *not* satisfy NT. Otherwise, if in (x, p) all items are assigned completely, then to show that (x, p) does *not* satisfy NT we have to show \neg (b). To this end consider the assignment x' and let $\delta_i = \sum_{j \in M} (x'_{i,j} - x_{i,j}) \alpha_j$ for $i \in N$, let $W = \{i \in N \mid \delta_i > 0\}$, and let $L = \{i \in N \mid \delta_i \leq 0\}$.

We begin by showing that $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq 0$.

For $i \in N$ we have $p'_i - p_i \leq \min(b_i - p_i, \delta_i v_i)$ because

$$\begin{aligned} p'_i \leq b_i &\Rightarrow p'_i - p_i \leq b_i - p_i, \\ u'_i \geq u_i &\Rightarrow p'_i - p_i \leq \delta_i v_i. \end{aligned} \quad \text{and}$$

It follows that

$$\begin{aligned} \sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i &\geq \sum_{i \in W} (p'_i - p_i) + \sum_{i \in L} (p'_i - p_i) \\ &= \sum_{i \in N} p'_i - \sum_{i \in N} p_i \geq 0. \end{aligned}$$

Next we show that $\sum_{i \in N} \delta_i v_i > 0$.

Since $u'_i \geq u_i$ for all $i \in N$ and $\sum_{i \in N} p'_i \geq \sum_{i \in N} p_i$ we have

$$\begin{aligned} \sum_{i \in N} u'_i \geq \sum_{i \in N} u_i &\Leftrightarrow \sum_{i \in N} (\sum_{j \in M} x'_{i,j} \alpha_j v_i - p'_i) \geq \sum_{i \in N} (\sum_{j \in M} x_{i,j} \alpha_j v_i - p_i) \\ &\Leftrightarrow \sum_{i \in N} (\sum_{j \in M} (x'_{i,j} - x_{i,j}) \alpha_j v_i) \geq \sum_{i \in N} p'_i - \sum_{i \in N} p_i \\ &\Rightarrow \sum_{i \in N} \delta_i v_i \geq \sum_{i \in N} p'_i - \sum_{i \in N} p_i \geq 0. \end{aligned} \quad (11)$$

If $u'_i > u_i$ for some $i \in N$, then $\sum_{i \in N} u'_i > \sum_{i \in N} u_i$ and, thus, the first inequality in (11) is strict. Otherwise, if $\sum_{i \in N} p'_i > \sum_{i \in N} p_i$, then the second inequality in (11) is strict. In both cases strictness of the inequality implies that $\sum_{i \in N} \delta_i v_i > 0$.

G Proof of Proposition 5

We begin by showing that if M satisfies VM and PI, then it satisfies IC. For a contradiction assume that M satisfies VM and PI, but that it does *not* satisfy IC. Then there exists $i \in N$, $\theta_i = (v_i, b_i)$, $\theta'_i = (v'_i, b_i)$, and $\theta_{-i} = (v_{-i}, b_{-i})$ with $v_i \neq v'_i$ such that

$$u_i(x_i(\theta'_i, \theta_{-i}), p(\theta'_i, \theta_{-i}), \theta_i) > u_i(x_i(\theta_i, \theta_{-i}), p(\theta_i, \theta_{-i}), \theta_i).$$

Let $c_{\gamma_t}(b_i, \theta_{-i}) \leq v_i \leq c_{\gamma_{t+1}}(b_i, \theta_{-i})$ and let $c_{\gamma_{t'}}(b_i, \theta_{-i}) \leq v'_i \leq c_{\gamma_{t'+1}}(b_i, \theta_{-i})$.

If $v_i > v'_i$ then since M satisfies VM and PI the utilities u_i and u'_i that agent i gets from reports θ_i and θ'_i satisfy

$$\begin{aligned} u_i - u'_i &= (\gamma_t - \gamma_{t'})v_i - \sum_{s=t'+1}^t (\gamma_s - \gamma_{s-1})c_{\gamma_s}(b_i, \theta_{-i}) \\ &\geq (\gamma_t - \gamma_{t'})v_i - \sum_{s=t'+1}^t (\gamma_s - \gamma_{s-1})v_i \\ &= 0. \end{aligned}$$

If $v_i < v'_i$ then since M satisfies VM and PI the utilities u'_i and u_i that agent i gets from reports θ'_i and θ_i satisfy

$$\begin{aligned} u'_i - u_i &= (\gamma_{t'} - \gamma_t)v_i - \sum_{s=t+1}^{t'} (\gamma_s - \gamma_{s-1})c_{\gamma_s}(b_i, \theta_{-i}) \\ &\leq (\gamma_{t'} - \gamma_t)v_i - \sum_{s=t+1}^{t'} (\gamma_s - \gamma_{s-1})v_i \\ &= 0. \end{aligned}$$

We conclude that in both cases agent i is weakly better off when he reports truthfully. This contradicts our assumption that M does *not* satisfy IC.

Next we show that if M satisfies IC, then it satisfies VM. By contradiction assume that M satisfies IC, but that it does *not* satisfy VM. Then there exists $i \in N$, $\theta_i = (v_i, b_i)$, $\theta'_i = (v'_i, b_i)$, and $\theta_{-i} = (v_{-i}, b_{-i})$ with $v_i < v'_i$ such that

$$\sum_{j \in M} x_{i,j}(\theta_i, \theta_{-i})\alpha_j > \sum_{j \in M} x_{i,j}(\theta'_i, \theta_{-i})\alpha_j.$$

Since M satisfies IC agent i with type θ_i does *not* benefit from reporting θ'_i , and vice versa. Thus,

$$\begin{aligned} \sum_{j \in M} x_{i,j}(\theta_i, \theta_{-i})\alpha_j v_i - p_i(\theta_i, \theta_{-i}) &\geq \sum_{j \in M} x_{i,j}(\theta'_i, \theta_{-i})\alpha_j v_i - p_i(\theta'_i, \theta_{-i}), \quad \text{and} \\ \sum_{j \in M} x_{i,j}(\theta'_i, \theta_{-i})\alpha_j v'_i - p_i(\theta'_i, \theta_{-i}) &\geq \sum_{j \in M} x_{i,j}(\theta_i, \theta_{-i})\alpha_j v'_i - p_i(\theta_i, \theta_{-i}). \end{aligned}$$

By combining these inequalities we get

$$\left(\sum_{j \in M} x_{i,j}(\theta_i, \theta_{-i})\alpha_j - \sum_{j \in M} x_{i,j}(\theta'_i, \theta_{-i})\alpha_j \right) (v_i - v'_i) \geq 0.$$

Since $\sum_{j \in M} x_{i,j}(\theta_i, \theta_{-i})\alpha_j > \sum_{j \in M} x_{i,j}(\theta'_i, \theta_{-i})\alpha_j$ this shows that $v_i \geq v'_i$ and gives a contradiction to our assumption that $v_i < v'_i$.

We conclude the proof by showing that if M satisfies IC, then it satisfies PI. For a contradiction assume that M satisfies IC, but that it does *not* satisfy PI. Then there exists $i \in N$, $\theta'_i = (v'_i, b_i)$, and $\theta_{-i} = (v_{-i}, b_{-i})$ with $c_{\gamma_{t'}} \leq v'_i \leq c_{\gamma_{t'+1}}$ such that

$$p_i(\theta'_i, \theta_{-i}) \neq p_i((0, b_i), \theta_{-i}) + \sum_{s=1}^{t'} (\gamma_s - \gamma_{s-1})c_{\gamma_s}(b_i, \theta_{-i}),$$

where the γ_s are the sum over the α 's of all possible assignments in non-increasing order and the $c_{\gamma_s}(b_i, \theta_{-i})$ are the smallest valuations (or critical valuations) that make agent i win γ_s .

Consider the smallest v'_i such that this is the case. For this v'_i we must have $v'_i = c_{\gamma_{t'}}(b_i, \theta_{-i}) > c_{\gamma_0}(b_i, \theta_{-i}) = 0$. We must have $v'_i = c_{\gamma_{t'}}(b_i, \theta_{-i})$ because by VM agent i 's assignment for all reports $\theta''_i = (v''_i, b_i)$ with v''_i such that $c_{\gamma_{t'}}(b_i, \theta_{-i}) \leq v''_i \leq c_{\gamma_{t'+1}}(b_i, \theta_{-i})$ is the same and, thus, by IC he must face the same payment. We must have $c_{\gamma_{t'}}(b_i, \theta_{-i}) > c_{\gamma_0}(b_i, \theta_{-i}) = 0$ because for $v'_i = 0$ we have $p(\theta'_i, \theta_{-i}) = p((0, b_i), \theta_{-i})$ by definition.

Case 1: $p_i(\theta'_i, \theta_{-i}) > p_i((0, b_i), \theta_{-i}) + \sum_{s=1}^{t'} (\gamma_s - \gamma_{s-1}) c_{\gamma_s}(b_i, \theta_{-i})$

Consider $\theta_i = (v_i, b_i)$ with $v_i < v'_i$ such that $c_{\gamma_{t'-1}}(b_i, \theta_{-i}) \leq v_i \leq c_{\gamma_{t'}}(b_i, \theta_{-i})$. Since $v_i < v'_i$ we have $p_i(\theta_i, \theta_{-i}) = p_i((0, b_i), \theta_{-i}) + \sum_{s=1}^{t'-1} (\gamma_s - \gamma_{s-1}) c_{\gamma_s}(b_i, \theta_{-i})$. If agent i 's type is θ'_i then for the utilities u'_i and u_i that he gets for reports θ'_i and θ_i we have

$$u'_i - u_i < (\gamma_{t'} - \gamma_{t'-1})v'_i - (\gamma_{t'} - \gamma_{t'-1})c_{\gamma_{t'}}(b_i, \theta_{-i}) = 0.$$

This shows that agent i with type θ'_i has an incentive to misreport his type as θ_i and contradicts our assumption that M satisfies IC.

Case 2: $p_i(\theta'_i, \theta_{-i}) < p_i((0, b_i), \theta_{-i}) + \sum_{s=1}^{t'} (\gamma_s - \gamma_{s-1}) c_{\gamma_s}(b_i, \theta_{-i})$

Let $\epsilon = p_i((0, b_i), \theta_{-i}) + \sum_{s=1}^{t'} (\gamma_s - \gamma_{s-1}) c_{\gamma_s}(b_i, \theta_{-i}) - p_i(\theta'_i, \theta_{-i})$ and consider $\theta_i = (v_i, b_i)$ with $v_i < v'_i$ such that $c_{\gamma_{t'-1}}(b_i, \theta_{-i}) \leq v_i \leq c_{\gamma_{t'}}(b_i, \theta_{-i})$. Since $v_i < v'_i$ we have $p_i(\theta_i, \theta_{-i}) = p_i((0, b_i), \theta_{-i}) + \sum_{s=1}^{t'-1} (\gamma_s - \gamma_{s-1}) c_{\gamma_s}(b_i, \theta_{-i})$. If agent i 's type is θ_i then for the utilities u'_i and u_i that he gets from reports θ'_i and θ_i we have

$$u'_i - u_i = (\gamma_{t'} - \gamma_{t'-1})v_i - (\gamma_{t'} - \gamma_{t'-1})c_{\gamma_{t'}}(b_i, \theta_{-i}) + \epsilon$$

Since this is true for all v_i with $c_{\gamma_{t'-1}}(b_i, \theta_{-i}) \leq v_i \leq c_{\gamma_{t'}}(b_i, \theta_{-i})$ we can choose v_i such that $(\gamma_{t'} - \gamma_{t'-1})(v_i - c_{\gamma_{t'}}(b_i, \theta_{-i})) > -\epsilon$. We get $u'_i - u_i > 0$. This shows that agent i with type θ_i has an incentive to misreport his type as θ'_i and contradicts our assumption that M satisfies IC.

H Proof of Proposition 6

First suppose that the payments are deterministic. If $p_i > b_i$ then $\tilde{p}_i > b_i$ and $u_i(x_i, p_i, (v_i, b_i)) = u_i(\tilde{x}_i, \tilde{p}_i, (\tilde{v}_i, b_i)) = -\infty$. Otherwise,

$$u_i(x_i, p_i, (v_i, b_i)) = \sum_{j=1}^m (x_{i,j} \alpha_j v_i) - p_i = \tilde{x}_i \tilde{v}_i - \tilde{p}_i = u_i(\tilde{x}_i, \tilde{p}_i, (\tilde{v}_i, b_i)).$$

Next suppose that the payments are randomized. If $\Pr[p_i > b_i] > 0$ then $\Pr[\tilde{p}_i > b_i] > 0$ and $\mathbb{E}[u_i(x_i, p_i, (v_i, b_i))] = \mathbb{E}[u_i(\tilde{x}_i, \tilde{p}_i, (\tilde{v}_i, b_i))] = -\infty$. Otherwise,

$$\mathbb{E}[u_i(x_i, p_i, (v_i, b_i))] = \mathbb{E}\left[\sum_{j=1}^m (x_{i,j} \alpha_j v_i) - p_i\right] = \mathbb{E}[\tilde{x}_i \tilde{v}_i - \tilde{p}_i] = \mathbb{E}[u_i(\tilde{x}_i, \tilde{p}_i, (\tilde{v}_i, b_i))].$$