On Multiple Keyword Sponsored Search Auctions with Budgets

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We study *multiple keyword* sponsored search auctions with budgets. Each keyword has *multiple ad slots* with a click-through rate. The bidders have additive valuations, which are linear in the click-through rates, and budgets, which are restricting their overall payments. Additionally, the number of slots per keyword assigned to a bidder is bounded.

We show the following results: (1) We give the first mechanism for multiple keywords, where click-through rates differ among slots. Our mechanism is incentive compatible in expectation, individually rational in expectation, and Pareto optimal. (2) We study the combinatorial setting, where each bidder is only interested in a subset of the keywords. We give an incentive compatible, individually rational, Pareto optimal, and deterministic mechanism for identical click-through rates. (3) We give an impossibility result for incentive compatible, individually rational, Pareto optimal, and deterministic mechanisms for bidders with diminishing marginal valuations.

1. INTRODUCTION

In *sponsored search* (or *adwords*) auctions advertisers bid on *keywords*. Such auctions are used by firms such as Google, Yahoo, and Microsoft [Edelman et al. 2005]. The search result page for each keyword contains multiple slots for ads and each bidder is assigned to a limited number of slots on a search result page. The slots have a click-through rate (CTR), which is usually decreasing by the position of the slot on the search result page. The true valuation of the bidders is private knowledge and is assumed to depend linearly on the CTR. Moreover, valuations are assumed to be additive, i.e., the total valuation of a bidder is equal to the sum of his valuations for all the slots that are assigned to him.

A further key ingredient of an adwords auction is that bidders specify a budget on the payment charged for the ads, effectively linking the different keywords. The deterministic Vickrey auction [Vickrey 1961] was designed to maximize social welfare in this and more general settings without budget restrictions. However, the introduction of budgets dramatically changes the nature of the problem. The Vickrey auction may charge more than the budget and is no longer feasible. Moreover, bidders might get assigned to slots even though their budget is arbitrary small and other bidders are interested in those slots. Thus, as was observed before [Dobzinski et al. 2008; 2011], maximizing social welfare is not the right optimality criterion to use. In a seminal paper by Dobzinski et al. [2008; 2011], they considered the multi-unit case with additive valuations, which in the sponsored search setting corresponds to each keyword having only one slot and all slots having identical CTR. They gave an incentive compatible (IC) auction based on Ausubel's ascending clinching auction [Ausubel 2004] that produces a Pareto optimal (PO) and individually rational (IR) allocation if budgets are *public*. They also showed that this assumption is strictly needed, i.e., that no deterministic mechanism for *private* budgets exists if we insist on incentive compatibility, individual rationality, and on obtaining an allocation that is Pareto optimal. This impossibility result for *deterministic* mechanisms was strengthened for our setting to *public* budgets in [Dütting et al. 2012]. The question was open what optimality result can be achieved for *randomized* mechanisms. Due to the impossibility results for deterministic mechanisms it is unlikely that "strong" optimality criteria, such as bidder optimality, are achievable. Thus, the first question to study is whether Pareto optimality, which is a basic notion of optimality, can be achieved with *randomized* mechanisms. Note that if an allocation is Pareto optimal then it is impossible to make a bidder better off without making another bidder or the auctioneer worse off, and is therefore the least one should aim for.

Our Results.. We give a positive answer to the above question and also present two further related results. Specifically, the paper contains the following three results: (1) *Multiple keywords with multiple slots:* We show that the multi-unit auction of Dobzinski et al. [2008; 2011] can be adapted to study adwords auctions with multiple keywords having multiple slots, and budget limits for each bidder. We specifically model the case of several slots with different CTR, available for each keyword, and a bound on the number of slots per keyword (usually one) that can be allocated to a bidder. We first provide an IC, IR, and PO deterministic auction that provides a fractional allocation for the case of one keyword with divisible slots. Note that the impossibility result in [Dütting et al. 2012] does not hold for divisible slots. In contrast, the impossibility result in [Dobzinski et al. 2008; 2011] for multi-unit auctions applies also to this setting, and achieving IC, IR, and PO deterministic auctions is only possible if budgets are *public*. Thus, we restrict ourselves to the public budget case. Our auction is a one-shot auction, i.e., each bidder interacts only once with the auction. We then show how to probabilistically round this fractional allocation for the divisible case to an integer allocation for the indivisible case with multiple keywords (i.e., the adwords setting) and get an auction that is IC in expectation, IR in expectation, and PO.

(2) *Multiple keywords with combinatorial constraints and multiple slots:* So far we assumed that every bidder is interested in every keyword. In the second part of the paper we study the case that bidders are interested in only a subset of the keywords, i.e., bidders have a non-zero identical valuation only on a *subset* of the keywords. The valuations are additive and each bidder is assigned at most one slot for a given keyword. We restrict the model by allowing only identical slots for each keyword, i.e., we require that all slots have the same CTR. This setting extends the combinatorial one-slot per keyword model considered by Fiat et al. [2011] to multiple slots. We present a variation of the clinching auction that is deterministic, IC, IR, and PO.

(3) Finally, we also study non-additive valuations, namely valuations with diminishing marginal valuations. Diminishing marginal valuation (also called submodular) functions are widely used to model auction settings with marginal utilities being positive functions that are non-increasing in the number of items already allocated to the bidders. We show that even in the multi-unit (one slot per keyword) case there is no deterministic, IC, IR, and PO auction for private diminishing marginal valuations and public budgets. This shows how budgets complicate mechanism design: For the non-budgeted version of this setting Ausubel [2004] gave his deterministic mechanism.

Related Work.. Ascending clinching auctions are used in the FCC spectrum auctions, see [Milgrom 2000; Ausubel and Milgrom 2002; Ausubel 2004]. For a motivation of adwords auctions see [Nisan et al. 2009] on Google's auction for TV ads.

We first compare our results with those of a recent, unpublished work by Goel et al. [2012] that was developed independently at the same time. They studied IC auctions with feasible allocations that must obey public polymatroid constraints and agents with identical or separable valuations (see their Lemma 3.10) and public budgets. The problem of auctions with polymatroid constraints was first studied by Bikhchandani et al. [2008] for unbudgeted bidders and concave utilities. The auction in [Goel et al. 2012] is an adaption of the ascending auction in [Bikhchandani et al. 2008] to the case of budgeted bidders. The polymatroid constraints generalize on one hand the the multi-unit case in [Dobzinski et al. 2008; 2011] and the multiple slots with different CTRs model presented in this paper. On the other hand, the PO ascending auction in [Goel et al. 2012] only returns allocations for divisible items whereas in Section 4 of this paper we demonstrate that these allocations can be rounded to *allocations for indivisible items* if we allow the auction to yield incentive compatibility in expectation. In Section 5, we present an IC, IR, and PO *deterministic* auction with feasible *allocations of indivisible slots* that obey matching constraints for the case of multiple identical slots.

There are three extensions of Dobzinski et al. [2008; 2011]: (1) Fiat et al. [2011] studied an extension to a combinatorial setting, where items are distinct and different bidders may be interested in different items. The auction presented in [Fiat et al. 2011] is IC, IR, and PO for additive valuations and single-valued bidders (i.e., every bidder does not distinguish between the keywords in his public interest set). This is a special case of our combinatorial setting in Section 5 with multiple keywords but only one slot per keyword. (2) Bhattacharya et al. [2009] dealt with private budgets, and gave an auction for one infinitely divisible item, where bidders cannot improve their utility by underreporting their budget. This leads to a randomized IC in expectation auction for one infinitely divisible item with both private valuations and budgets. (3) Several papers [Aggarwal et al. 2009; Ashlagi et al. 2010; Dütting et al. 2011; Fujishige and Tamura 2007] studied *envy-free* outcomes that are bidder optimal, respectively PO, in an one-keyword adwords auction. In this setting they give (under certain conditions on the input) an IC auction with both private valuations and budgets.

Our impossibility result in Section 6 is related to two impossibility results: Lavi and May [2011] show that there is no IC, IR, and PO deterministic mechanism for indivisible items and bidders with *monotone* valuations. Our result for indivisible items is stronger as it applies to bidders with non-negative and diminishing marginal valuations. In [Goel et al. 2012] the same impossibility result for *divisible* items and bidders with monotone and concave utility functions was given. Note that neither their result nor ours implies the other.

2. PROBLEM STATEMENT AND DEFINITIONS

We have n bidders and m slots. We call the set of bidders $I := \{1, \ldots, n\}$ and the set of slots $J := \{1, \ldots, m\}$. Each bidder $i \in I$ has a private *valuation* v_i, a public *budget* b_i, and a public *slot constraint* κ_i, which is a positive integer. Each slot $j \in J$ has a public *quality* $\alpha_j \in \mathbb{Q}_{\geq 0}$. The slots are ordered such that $\alpha_j \geq \alpha_{j'}$ if $j > j'$, where ties are broken in some arbitrary but fixed order. We assume in Section 3 and 4 that the number of slots m fulfills $m = \sum_{i \in I} \kappa_i$ as the general case can be easily reduced to this setting.

Divisible case: In the divisible case we assume that there is only one keyword with infinitely divisible slots. Thus the goal is to assign each bidder i a fraction $x_{i,j} \geq 0$ of each slot j and charge him a payment p_i. A matrix $X = (x_{i,j})_{(i,j) \in I \times J}$ and a payment vector p are called an *allocation* (X, p). We call $c_i = \sum_{j \in J} \alpha_j x_{i,j}$ the *weighted capacity* allocated to bidder i. An allocation is *feasible* if it fulfills the following conditions: (1) the sum of the fractions assigned to a bidder does not exceed his *slot constraint* ($\sum_{j \in J} x_{i,j} \leq \kappa_i \ \forall i \in I$); (2) each of the slots is fully assigned to the bidders ($\sum_{i \in I} x_{i,j} = 1 \ \forall j \in J$); and (3) the payment of a bidder does not exceed his budget limit ($b_i \geq p_i \ \forall i \in I$).

Indivisible case: We additionally have a set R of *keywords*, where $|R|$ is public. The goal is to assign each slot $j \in J$ of keyword $r \in R$ to one bidder $i \in I$ while obeying various constraints. An assignment $X = (x_{i,j,r})_{(i,j,r) \in I \times J \times R}$ where $x_{i,j,r} = 1$ if slot j is assigned to bidder i in keyword r, and $x_{i,j,r} = 0$ otherwise, and a payment vector p form an *allocation* (X, p). We call $c_i = \sum_{j \in J} \frac{\alpha_j}{|R|} (\sum_{r \in R} x_{i,j,r})$ the *weighted capacity* allocated to bidder i. An allocation is *feasible* if it fulfills the following conditions: (1) the number of slots of a keyword that are assigned to a bidder does not exceed his *slot constraint* ($\sum_{j \in J} x_{i,j,r} \leq \kappa_i \; \forall i \in I, \forall r \in R$); (2) each slot is assigned to exactly one bidder ($\sum_{i \in I} x_{i,j,r} = 1 \; \forall j \in J, \forall r \in R$); and (3) the payment of a bidder does not exceed his budget limit ($b_i \geq p_i \; \forall i \in I$).

Combinatorial indivisible case: In the combinatorial case not all keywords are identical. Every bidder $i \in I$ has a publicly known set of interest $S_i \subseteq R$, and valuation v_i for all keywords in S_i and a valuation of zero for all other keywords. We model this case by imposing $x_{i,j,r} = 0 \; \forall r \notin S_i$.

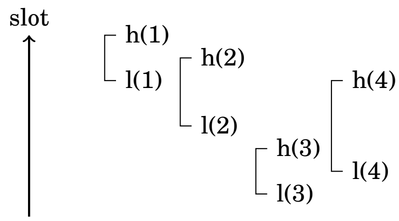
Note that in all cases the budgets are bounds on *total* payments across keywords and *not* bounds on prices of individual keywords.

Properties of the auctions: The *utility* u_i of bidder i for a feasible allocation (X, p) is $c_i v_i - p_i$, the *utility* of the auctioneer (or mechanism) is $\sum_{i \in I} p_i$. We study auctions that select feasible allocations obeying the following conditions: (*Bidder rationality*) $u_i \geq 0$ for all bidders $i \in I$, (*Auctioneer rationality*) the utility of the auctioneer fulfills $\sum_{i \in I} p_i \geq 0$, and (*No-positive-transfer*) $p_i \geq 0$ for all bidders $i \in I$. An auction that on all inputs outputs an allocation that is both bidder rational and auctioneer rational is called *individually rational (IR)*. A feasible allocation (X, p) is *Pareto optimal* (PO) if there is no other feasible allocation (X', p') such that (1) the utility of no bidder in (X, p) is less than his utility in (X', p'), (2) the utility of the auctioneer in (X, p) is no less than his utility in (X', p'), and (3) at least one bidder or the auctioneer is better off in (X', p') compared with (X, p). An auction is *incentive compatible* (IC) if it is a dominant strategy for all bidders to reveal their true valuation. An auction is said to be *Pareto optimal* (PO) if the allocation it produces is PO. A randomized auction is IC in expectation, IR in expectation, respectively PO in expectation if the above conditions hold in expectation. We show that our randomized mechanism for indivisible slots is PO in expectation and that each realized allocation is PO. Note that neither of these conditions implies the other (see Appendix A).

3. DETERMINISTIC CLINCHING AUCTION FOR THE DIVISIBLE CASE

3.1. Characterization of Pareto Optimality

In this section we present a novel characterization of PO allocations that allows to address the divisible case of multiple slots with different CTRs. Given a feasible allocation (X, p), a *swap* between two bidders i and i' is a fractional exchange of slots, i.e., if there are slots j and j' and a constant $\tau > 0$ with $x_{i,j} \geq \tau$ and $x_{i',j'} \geq \tau$ then a swap between i and i' gives a new feasible (X', p) with $x'_{i,j} = x_{i,j} - \tau$, $x'_{i',j'} = x_{i',j'} - \tau$, $x'_{i,j'} = x_{i,j'} + \tau$, and $x'_{i',j} = x_{i',j} + \tau$. If $\alpha_j < \alpha_{j'}$ then the swap increases i's weighted capacity. We assume throughout this section that $\alpha_j \neq \alpha_{j'}$ for $j \neq j'$, the general case requires a small modification presented in the full version of our paper. To characterize PO allocations we first define for each bidder i the set N_i of bidders such that for every bidder a in N_i there exists a swap between i and a that increases i's weighted capacity. Given a feasible allocation (X, p) we use $h(i) := \max\{j \in J | x_{i,j} > 0\}$ for the slot with the highest quality that is assigned to bidder i and $l(i) := \min\{j \in J | x_{i,j} > 0\}$ for the slot with the lowest quality that is assigned to bidder i. Now, $N_i = \{a \in I | h(a) > l(i)\}$ is the set of all the bidders a such that i could increase his weighted capacity (and a could decrease his weighted capacity) if i traded with a, for example, if i received part of a's share of slot $h(a)$. To model sequences of swaps we define furthermore $N_i^k = N_i$ for $k = 1$ and $N_i^k = \bigcup_{a \in N_i^{k-1}} N_a$ for $k > 1$. Since we have only n bidders, $\bigcup_{k=1}^{n} N_i^k = \bigcup_{k=1}^{n'} N_i^k$ for all $n' \geq n$. We define $\tilde{N}_i := \bigcup_{k=1}^{n} N_i^k \setminus \{i\}$ as the set of *desired (recursive) trading partners* of i. See Figure 3.1 for an example with four bidders. The bidders a in $\tilde{N}_i$ are all the bidders such that through a sequence of trades that "starts"



$N_1 = \{1,2\}$ $\quad N_2 = \{1,2,4\}$
$N_3 = \{1,2,3,4\}$ $\quad N_4 = \{1,2,3,4\}$

$N_1^2 = \{1,2,4\}$ $\quad N_2^2 = \{1,2,3,4\}$

$N_1^3 = \{1,2,3,4\}$

$\tilde{N}_1 = \{2,3,4\}$ $\quad \tilde{N}_2 = \{1,3,4\}$
$\tilde{N}_3 = \{1,2,4\}$ $\quad \tilde{N}_4 = \{1,2,3\}$

Fig. 1. Example of desired trading partners

with i and "ends" with a, bidder i could increase his weighted capacity, bidder a could decrease his weighted capacity, and the capacity of the remaining bidders involved in the swap would been unchanged. Now let $\tilde{v}_i = \min_{a \in \tilde{N}_i}(v_a)$ if $\tilde{N}_i \neq \emptyset$ and $\tilde{v}_i = \infty$ else.

Given a feasible allocation (X,p) we use $B := \{i \in I | b_i > p_i\}$ to denote the set of bidders who have a positive remaining budget. As we show below if for a given assignment we know $\tilde{v}_i$ for every bidder $i \in B$ then we can immediately decide whether the assignment is PO or not.

We say that a feasible allocation (X,p) contains a *trading swap sequence* (for short *trading swap*) if there exists a feasible allocation (X',p') and two bidders $u,w \in I$ such that

(1) bidder w is a desired trading partner of u, i.e., $w \in \tilde{N}_u$,
(2) for all $i \in I \setminus \{u,w\}$ it holds that the weighted capacity of i and the payment of i are unchanged by the swap, i.e., $\sum_{j \in J} \alpha_j x_{i,j} = \sum_{j \in J} \alpha_j x'_{i,j}$ and $p_i = p'_i$,
(3) the weighted capacity of u increases by $\delta > 0$ and the weighted capacity of w decreases by δ, i.e., $\delta := \sum_{j \in J} \alpha_j (x'_{u,j} - x_{u,j}) = \sum_{j \in J} \alpha_j (x_{w,j} - x'_{w,j}) > 0$,
(4) $v_u > v_w$, u pays after the swap exactly that amount more that w's weighted valuation decreases (i.e., $p'_u - p_u = v_w \delta$), and w pays exactly that amount less (i.e., $p_w - p'_w = v_w \delta$), and
(5) u has a high enough budget to pay what is required by (X',p'), i.e., $b_u \geq p'_u$.

We say that the allocation (X',p') *results from the trading swap*. The existence of a trading swap is related to the $\tilde{v}_i$ of each bidder i with remaining budget.

THEOREM 3.1. *A feasible allocation (X,p) contains no trading swaps if and only if $\tilde{v}_i \geq v_i$ for each bidder $i \in B$.*

The following theorem shows that the absence of trading swaps characterizes Pareto optimality. We will use exactly this fact to prove that the mechanism of the next section outputs a PO allocation.

THEOREM 3.2. *A feasible allocation (X,p) is Pareto optimal if and only if it contains no trading swaps.*

Hence, the feasible allocation (X,p) is PO if and only if $\tilde{v}_i \geq v_i \ \forall i \in B$. This novel characterization of Pareto optimality is interesting, as the payment does not affect the values $\tilde{v}_i$, the payment only influences which bidders belong to B.

3.2. Multiple Keyword Auction for the Divisible Case

We describe next our deterministic clinching auction for divisible slots and show that it is IC, IR, and PO. The auction repeatedly increases a price "per capacity" and gives different weights to different slots depending on their CTRs. To perform the check whether all remaining unsold weighted capacity can still be sold we solve suitable linear programs. We will show that if the allocation of the auction did not fulfill the characterization of Pareto optimality given in Section 3.1, i.e., if it contained a trading swap, then one of the linear programs solved by the auction would not have computed an optimal solution. Since this is not possible, it will follow that the allocation is PO. A formal description of the auction is given in the procedures AUCTION and SELL. The input values of AUCTION are the bids, budget limits, and slot constraints that the bidders communicate to the auctioneer on the beginning of the auction, and information about the qualities of the slots. The auction is a so called "one-shot auction", the bidders are asked once for the valuations at the beginning of the auction and then they *cannot* input *any* further data.

Algorithm 1 Clinching auction for divisible slots

procedure AUCTION($I, J, \alpha, \kappa, v, b$)
  $A \leftarrow I;\ \pi \leftarrow 0;\ \pi^+ \leftarrow 1$
  $c_i \leftarrow 0,\ p_i \leftarrow 0,\ d_i \leftarrow \infty\ \forall i \in I$
  while $\sum_{i\in I} c_i < \sum_{j\in J} \alpha_j$ **do** `\\ unsold weighted capacity exists`
    $E \leftarrow \{i \in A | \pi^+ > v_i\}$ `\\ bidders become exiting bidders`
    for $i \in E$ **do**
      $(X, s) \leftarrow$ SELL($I, J, \alpha, \kappa, c, d, i$) `\\ sell to exiting bidder`
      $(c_i, p_i, d_i) \leftarrow (c_i + s, p_i + s\pi, 0)$
    end for
    $A \leftarrow A \setminus E$ `\\ exiting bidders leave auction`
    $d_i^+ \leftarrow \frac{b_i - p_i}{\pi^+}\ \forall i \in A$
    while $\exists i \in A$ with $d_i \neq d_i^+$ **do** `\\ bidders with price` π `exist`
      $i' \leftarrow \min(\{i \in A | d_i \neq d_i^+\})$ `\\ select bidder with price` π
      $(X, s) \leftarrow$ SELL($I, J, \alpha, \kappa, c, d, i'$) `\\ sell to bidder`
      $(c_{i'}, p_{i'}) \leftarrow (c_{i'} + s, p_{i'} + s\pi)$
      $d_{i'}^+ \leftarrow \frac{b_{i'} - p_{i'}}{\pi^+};\ d_{i'} \leftarrow d_{i'}^+$ `\\ increase bidder's price to` π^+
    end while
    $\pi \leftarrow \pi^+;\ \pi^+ \leftarrow \pi^+ + 1$ `\\ increase price`
  end while
  return (X, p)
end procedure

Algorithm 2 Determination of the weighted capacity that bidder i' clinches

procedure SELL($I, J, \alpha, \kappa, c, d, i'$)
  compute an optimal solution of the following linear program
  that is a vertex of the polytope defined by its constraints:

minimize		$\gamma_{i'}$	
s.t.:	(a)	$\sum_{i\in I} x_{i,j} = 1 \quad \forall j \in J$	▷ assign all slots
	(b)	$\sum_{j\in J} x_{i,j} = \kappa_i \quad \forall i \in I$	▷ slot constraint
	(c)	$\sum_{j\in J} x_{i,j}\, \alpha_j - \gamma_i = c_i \quad \forall i \in I$	▷ assign value to γ_i
	(d)	$\gamma_i \leq d_i \quad \forall i \in I$	▷ demand constraint
	(e)	$x_{i,j} \geq 0 \quad \forall i \in I, \forall j \in J$	
	(f)	$\gamma_i \geq 0 \quad \forall i \in I$	

  return $(X, \gamma_{i'})$
end procedure

The demand of the bidders for weighted capacity is computed by the mechanism based on their remaining budget and the current price. We assume throughout this section that $v_i \in \mathbb{N}_+$ and $b_i \in \mathbb{Q}_+$ for all $i \in I$.[1] The state of the auction is defined by the current price π, the next price π^+, the weighted capacity c_i that bidder $i \in I$ has clinched so far, and the payment p_i that has been charged so far to bidder i. We define the set of *active* bidders $A \subseteq I$ which are all those $i \in I$ with $\pi \leq v_i$, and the subset E of A of *exiting* bidders which are all those $i \in A$ with $\pi^+ > v_i$. The auction does not increase the price that a bidder $i \in I$ has to pay from π to π^+ for *all* bidders at the same time. Instead, it calls SELL each time before it increases the price for a single bidder. If the price that bidder $i \in A$ has to pay for weighted capacity is π then his demand is $d_i = \frac{b_i - p_i}{\pi}$. If the price he has to pay was already increased to π^+ then his demand is $d_i = \frac{b_i - p_i}{\pi^+} < \frac{b_i - p_i}{\pi}$. In this case, the demand corresponds to d_i^+, that is always equal to $\frac{b_i - p_i}{\pi^+}$. Different from the auction in [Dobzinski et al. 2008; 2011; Bhattacharya et al. 2009] a bidder with $d_i = d_i^+$ is also charged the increased price π^+ if he receives additional weighted capacity. Since our price is incremented by one in each round and is not continuously increasing as in prior work, this is necessary for proving the Pareto optimality of the allocation.

The crucial point of the auction is that it sells only weighted capacity s to bidder i at a certain price π or π^+ if it cannot sell s to the other bidders. It computes s by solving a linear program in SELL. We use a linear program as there are two types of constraints to consider: The slot constraint in line (b) of the LP, which constraints "unweighted" capacity, and the demand constraint in line (d) of the LP, which is implied by the budget limit and constraints weighted capacity. In the homogeneous item setting in [Dobzinski et al. 2008; 2011; Bhattacharya et al. 2009] there are no slot constraints and the demand constraints are unweighted, i.e., $\alpha_j = 1 \, \forall j \in J$. Thus, no linear program is needed to decide what amount to sell to whom.

For each iteration of the outer while-loop the auction first calls SELL for each exiting bidder i and sells him s for price π. This is the last time when he can gain weighted capacity. Afterward, he is no longer an active bidder. Next, it calls SELL for one of the remaining active bidders who has $d_i \neq d_i^+$. It sells him the respective s and increases his price to π^+. It continues the previous step until the price of each active bidder is increased to π^+. Then it sets π to π^+ and π^+ to $\pi^+ + 1$. To illustrate the mechanism we give an example in Appendix D.

It is crucial for the progress and the correctness of the mechanism that there is a feasible solution for the linear program in SELL every time that SELL is called. This is proved in Appendix E. It follows that the final assignment X is a feasible solution of the linear program in SELL. Thus it fulfills conditions (1) and (2) for a feasible allocation. Condition (3) is also fulfilled as by the definition of the demand of a bidder, the auction guarantees that $b_i \geq p_i$ for all i. Thus, the allocation (X, p) computed by the auction is a feasible allocation. As no bidder is assigned weighted capacity if his price is above his valuation and the mechanism never pays the bidders, the auction is IR. As it is an increasing price auction, it is also IC.

PROPOSITION 3.3. *The auction is individually rational and incentive compatible and the allocation (X, p) it outputs has only rational entries.*

We show finally that the allocation (X, p) our auction computes does not contain any trading swap, and thus, by Theorem 3.2 it is PO. The proof shows that every trading swap in (X, p) would lead to a superior solution to one of the linear programs solved

[1]All the arguments go through if we simply assume that $v_i \in \mathbb{Q}_+$ for all $i \in I$ and there exists a publicly known value $z \in \mathbb{R}_+$ such that for all bidders i and i' either $v_i = v_{i'}$ or $|v_i - v_{i'}| \geq z$.

by the mechanism. Since the mechanism found an optimal solution this leads to a contradiction.

THEOREM 3.4. *The allocation (X, p) returned by our auction is Pareto optimal.*

4. RANDOMIZED CLINCHING AUCTION FOR THE INDIVISIBLE CASE

We will now use the allocation computed by the deterministic auction for *divisible* slots to give a randomized auction for multiple keywords with *indivisible* slots that ensures that bidder i receives at most κ_i slots for each keyword. The randomized auction has to assign to every slot $j \in J$ exactly one bidder $i \in I$ for each keyword $r \in R$. We call a distribution over allocations for the indivisible case *Pareto superior* to another such distribution if the expected utility of a bidder or the auctioneer is higher, while all other expected utilities are at least as large. If a distribution has no Pareto superior distribution, we call it *Pareto optimal*. The basic idea is as follows: Given the PO solution for the *divisible* case, we construct a *distribution over allocations* of the *indivisible* case such that the expected utility of every bidder and of the auctioneer is the same as the utility of the bidder and the auctioneer in the *divisible* case. To be precise, we do not explicitly construct this distribution but instead we give an algorithm that can sample from this distribution. The mechanism for the *indivisible* case would, thus, first call the mechanism for the *divisible* case (with the same input) and then convert the resulting allocation (X^d, p^d) into a representation of a PO distribution over allocations for the *indivisible* case. It then samples from this representation to receive the allocation that it outputs. During all these steps the (expected) utility of the bidders and the auctioneer remains unchanged. As the mechanism for the *divisible* case is IR and IC this implies immediately that the mechanism for the *indivisible* case is IR in expectation and IC in expectation. To show that the final allocation is PO in expectation and also PO ex post we use the following lemma.

LEMMA 4.1. *For every probability distribution over feasible allocations in the indivisible case there exists a feasible allocation (X^d, p^d) in the divisible case, where the utility of the bidders and the auctioneer equals their expected utility using this probability distribution.*

Lemma 4.1 implies that any probability distribution over feasible allocations in the *indivisible* case that is Pareto superior to the distribution generated by our auction would lead to a feasible allocation for the *divisible* case that is Pareto superior to (X^d, p^d). This is not possible as (X^d, p^d) is PO. Additionally, each realized allocation is ex-post Pareto optimal: if in the *indivisible* case there existed a Pareto superior allocation to one of the allocations that gets chosen with a positive probability in our auction, then a Pareto superior allocation would exist in the *divisible* case. By the same argument as above this would lead to a contradiction.

We still need to explain how to use the PO allocation (X^d, p^d) for the *divisible* case to give a probability distribution for the *indivisible* case with expected utility for every bidder equal to the utility in the divisible case and how to sample efficiently from this distribution. Given an input for the indivisible case we use it *as is* as an input for the algorithm for the divisible case, ignoring the number of keywords. Based on the allocation (X^d, p^d) for the divisible problem we construct a matrix M' of size $|J| \times \lambda$, where λ is the least common denominator of all the $x^d_{i,j}$ values and where each column of M' corresponds to a feasible assignment for the indivisible one-keyword case. Note that the same assignment can occur in multiple columns of M'. The matrix M' is our representation of the distribution over allocations in the indivisible case. To sample from the distribution we pick for each $r \in R$ a column uniformly at random from the columns of M'. The r-th choice gives the assignment of bidders to the slots of keyword r. The payments are set equal to p^d. We give in Appendix I the construction of M' such that after the above sampling step the expected weighted capacity allocation to bidder $i \in I$ equals $\sum_{j\in J} \alpha_j x^d_{i,j}$, i.e., its weighted capacity in the divisible case. Additionally, all of the slots are fully assigned to the bidders, and hence, the stated properties are fulfilled by the randomized auction. An example for the randomization is given in Appendix J.

5. THE COMBINATORIAL CASE WITH MULTIPLE SLOTS

We consider single-valued combinatorial auctions with multiple identical slots in multiple keywords. Every bidder $i \in I$ has valuation v_i on all keywords of his interest set S_i. All other keywords are valued zero. The interest sets S_i and the budgets b_i are public knowledge. We further restrict to the case where at most one slot per keyword is allocated to a single bidder (i.e., $\kappa_i = 1$). We require that at least m bidders are interested in each keyword, where m is the number of slots for a keyword.

In our auction, we extend the techniques of Fiat et al. [2011] for their single-slot per keyword setting to our multi-slot per keyword setting as follows: (1) We extend their B-matchings based approach by giving capacities, equal to the number of unsold slots, to vertices that represent keywords. (2) We extend the concept of trading alternating paths in the bidder/keyword bipartite graph, which in turn allows us to give a characterization of Pareto optimality for the multi-slot case. While in the single-slot case it is sufficient to restrict the attention to simple trading alternating paths, in our case there might be trading options where the same bidder or item can appear many times along the same path. The crucial insight is that there always exists a simple trading path whenever there exists a non-simple one.

We characterize a feasible allocation (H, p) by a tuple $H = (H_1, H_2, \ldots, H_n)$, where $H_i \subseteq S_i$ represents the set of keywords that are allocated to bidder i, and by a vector of payments $p = (p_1, p_2, \ldots, p_n)$ with $p_i \leq b_i$ for all $i \in I$. The utility of bidder i is defined by $u_i := v_i|H_i| - p_i$, and the utility of the auctioneer is $\sum_{i=1}^{n} p_i$. We base the allocation of the items in the clinching auction on B-matchings computed on a bipartite graph G with the union of keywords and bidders ($I \cup R$) as vertex set and the preferences $\{(i, t) \in I \times R | t \in S_i\}$ as edge set. The vertices have degree constraints, which represent the demand constraints for the bidders and the number of unsold slots for the keywords. The B-matchings are the subgraphs of G, which fulfill the constraints, and have a maximal number of edges. The idea of the auction is to sell slots at the highest possible price such that all slots are sold and there exists no competition between bidders. On the contrary, the existence of a trading path indicates that there exists competition on the assignment of the first slot in the path. We define the auction in Algorithm 4 and prove the following theorem in Appendix K.

THEOREM 5.1. *The allocation (H^*, p^*) produced by Algorithm 4 is incentive compatible, individually rational, and Pareto optimal.*

6. IMPOSSIBILITY FOR DIMINISHING MARGINAL VALUATIONS

We assume in this section that we have multiple homogeneous indivisible items and bidders with private diminishing marginal valuations and public budgets. We show that there is no IC, IR, and PO deterministic mechanism for this case.

Bidder i's marginal valuation for obtaining a further item when k items are already assigned to him is $v_i(k + 1)$. His valuation for obtaining $k + 1$ items is therefore $\sum_{j=1}^{k+1} v_i(j)$. The marginal valuations have to fulfill $v_i(k) \geq v_i(k+1)$ for $k \geq 1$. The initial clinching auction in [Ausubel 2004] was indeed proposed for the case of diminishing marginal valuations but without budget limits.

We use that the case of additive valuations, which was studied by Dobzinski et al. [2008; 2011], is a special case of ours, and that they showed that their auction is the only IC, IR, and PO deterministic auction for that case. We study bidders with diminishing marginal valuations that report additive valuations in order to raise the price paid by the other bidders and consequently decrease their demand. A possible decrease of the price charged to the non-truth telling bidders follows.

THEOREM 6.1. *There is no incentive compatible, individually rational, Pareto optimal, and deterministic mechanism for multiple homogeneous indivisible items and agents with private diminishing marginal valuations and public budget limits.*

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A. PARETO OPTIMALITY IN EXPECTATION

Let us assume that we have two bidders, a single indivisible item, and a uniformly distributed random variable $Y \sim \mathcal{U}(0,1)$. Consider first the case that bidder 1 has valuation $v_1 = 1$ and budget $b_1 = 1$, bidder 2 has valuation $v_2 = 2$ and budget $b_2 = 1$, and we have a value $\tilde{y} \in (0,1)$. If we sell the item to the bidder 2 for price $p_2 = 1$ (and $p_1 = 0$) for every realization y of Y with $y \neq \tilde{y}$ the allocation is PO in expectation. However, only if we sell the item to bidder 2 also for $y = \tilde{y}$ every possibly realized allocation is PO. Hence, PO in expectation does not imply that each realized allocation is PO. Next consider the case that bidder 1 has valuation $v_1 = 1$ and budget $b_1 = 1$, and bidder 2 has valuation $v_2 = 2$ and budget $b_2 = 0.5$. If we sell the item to bidder 1 for price $p_1 = 1$ (and $p_2 = 0$) for every realization $y \in (0,1)$ each realized allocation is PO because $v_1 > b_2$. However, we could select the bidder who gets the item with probability one half, and both bidders have to pay $p_1 = p_2 = 0.5$ independent of the assignment. Hence, the allocation is not PO in expectation, and therefore, PO in expectation is not implied if every realized allocation is PO.

B. PROOF OF THEOREM 3.1

We show first that given a feasible allocation (X,p) that contains a trading swap, there exists a bidder $u \in B$ with $\tilde{v}_u < v_u$. This direction follows directly from the definition of a trading swap. The valuation of u's desired trading partner w is $v_w \geq \tilde{v}_u$ and we know that $v_u > v_w$. Thus, $v_u > \tilde{v}_u$.

We will show now the other direction, i.e., that given a feasible allocation (X,p) such that $\exists u \in B: \ \tilde{v}_u < v_u$ it exists a trading swap in (X,p). We know that there is a bidder $u \in B$ with $\tilde{v}_u < v_u$. Thus, we can select the smallest $k \in \{1,\ldots,n\}$ for which there is a bidder $a_k \in N_u^k$ who has $v_{a_k} = \tilde{v}_u$. We define for all $p \in \{1,\ldots,k-1\}$ the bidder a_p such that $a_p \in N_u^p$ and $a_{p+1} \in N_{a_p}$ and set $a_0 := u$. Since we selected the smallest k, we know that $a_p \neq a_{p'}$ if $p \neq p'$. The fact that $a_{p+1} \in N_{a_p}$ implies that $h(a_{p+1}) > l(a_p)$. Hence, we could swap a fraction of size $\epsilon_{p+1} := \min\{x_{a_p,l(a_p)}, x_{a_{p+1},h(a_{p+1})}\}$ of the slots $h(a_{p+1})$ and $l(a_p)$ between the bidders a_{p+1} and a_p with $p \in \{0,\ldots,k-1\}$. Such a swap increases the weighted capacity that is assigned to bidder a_p by $\delta_{p+1} := \epsilon_{p+1}(\alpha_{h(a_{p+1})} - \alpha_{l(a_p)})$, while the weighted capacity that is assigned to bidder a_{p+1} is decreased by δ_{p+1}. We define $\delta := \min(\{\frac{b_{a_0} - p_{a_0}}{v_{a_k}}\} \cup \{\delta_p | p \in \{1,\ldots,k\}\})$ and $\tau_{p+1} := \frac{\delta}{\alpha_{h(a_{p+1})} - \alpha_{l(a_p)}}$ $\forall p \in \{0,\ldots,k-1\}$ and define an allocation (X',p') as follows: We set $x'_{a_p,h(a_{p+1})} := x_{a_p,h(a_{p+1})} + \tau_{p+1}$ and $x'_{a_p,l(a_p)} := x_{a_p,l(a_p)} - \tau_{p+1}$ for all $p \in \{0,\ldots,k-1\}$, $x'_{a_p,h(a_p)} := x_{a_p,h(a_p)} - \tau_p$ and $x'_{a_p,l(a_{p-1})} := x_{a_p,l(a_{p-1})} + \tau_p$ for all $p \in \{1,\ldots,k\}$, and $x'_{i,j} = x_{i,j}$ for all other $(i,j) \in I \times J$. Moreover, we set $p'_{a_k} := p_{a_k} - v_{a_k}\delta$, $p'_{a_0} := p_{a_0} + v_{a_k}\delta$, and $p'_i := p_i$ for all other $i \in I$. Thus, with $w = a_k$ it follows that (X',p') fulfills conditions (1)-(5) of a trading swap.

Next we show that (X',p') is a feasible allocation. By the definition of X' for all $i \in I$ it holds that $\sum_{j \in J} x'_{i,j} = \sum_{j \in J} x_{i,j} = \kappa_i$ as whenever for some τ with $-1 \leq \tau \leq 1$, $x'_{i,j}$ is set to $x_{i,j} + \tau$ for some $j \in J$, $x'_{i,l}$ is set to $x_{i,l} - \tau$ for some other $l \in J$. Additionally for every $j \in J$ it holds that $\sum_{i \in I} x'_{i,j} = \sum_{i \in I} x_{i,j} = 1$ as whenever $x'_{a_p,j}$ is set to $x_{a_p,j} + \tau$ for some τ with $-1 \leq \tau \leq 1$, either $x'_{a_{p+1},j}$ is set to $x_{a_{p+1},j} - \tau$, or $x'_{a_{p-1},j}$ is set to $x_{a_{p-1},j} - \tau$. Finally, $p'_i \leq p_i \leq b_i$ for all $i \neq u$ and by our construction $p'_u \leq b_u$. This shows that conditions (1) - (3) of a feasible allocation hold for (X',p').

C. PROOF OF THEOREM 3.2

We assume without loss of generality that the bidders are ordered by their valuation, i.e., $v_i \geq v_{i'}$ if $i > i'$.

The allocation that results from a trading swap in (X, p) is Pareto superior to (X, p). Hence, a Pareto optimal allocation cannot contain a trading swap. That proves the one direction.

The following part of the proof shows the other direction of the theorem, i.e., (X, p) is PO if it contains no trading swap. We use Theorem 3.1 and show instead that if $\tilde{v}_i \geq v_i \; \forall i \in B$ then there exists no Pareto superior allocation (X', p'). Let us assume that we have a feasible allocation (X', p') that is Pareto superior to (X, p). The utility of the auctioneer does not decrease. Thus, the sum of the payments of the bidders fulfills $\sum_{i \in I} p'_i \geq \sum_{i \in I} p_i$. If $\sum_{i \in I} p'_i > \sum_{i \in I} p_i$ then an allocation (X', p'') where $\sum_{i \in I} p''_i = \sum_{i \in I} p_i$ exists, which is Pareto superior compared to (X, p) as well: simply give the additional payments back to some of the bidders. Therefore, it suffices to consider the case where $\sum_{i \in I} p'_i = \sum_{i \in I} p_i$.

Let $q_i = \sum_{j \in J} \alpha_j (x'_{i,j} - x_{i,j})$ be the weighted capacity change of bidder i. Since (X, p) and (X', p') are feasible allocations, $\sum_{i \in I} x_{i,j} = 1$ for all $j \in J$, and $\sum_{i \in I} x'_{i,j} = 1$ for all $j \in J$. Hence, $\sum_{i \in I} q_i = \sum_{i \in I} \sum_{j \in J} \alpha_j (x'_{i,j} - x_{i,j}) = \sum_{j \in J} \alpha_j (\sum_{i \in I} x'_{i,j} - \sum_{i \in I} x_{i,j}) = 0$. It follows that (a) $\sum_{b \in I: q_b \leq 0} (-q_b) = \sum_{i \in I: q_i > 0} q_i$. As $\sum_{i \in I} p_i = \sum_{i \in I} p'_i$ it also follows that (b) $\sum_{b \in I: q_b \leq 0} (p_b - p'_b) = \sum_{i \in I: q_i > 0} (p'_i - p_i)$.

We partition the bidders into the following three sets: $I^- = \{b \in I | q_b \leq 0\}$, $B^+ = \{i \in B | q_i > 0\}$, and $C^+ = \{i \in I \setminus B | q_i > 0\} = I \setminus (I^- \cup B^+)$. We will show below that (A) $\sum_{b \in I^-} (p_b - p'_b) \geq \sum_{b \in I^-} (-q_b v_b) \geq \sum_{i \in B^+} q_i v_i$, (B) $\sum_{i \in B^+} q_i v_i \geq \sum_{i \in B^+} (p'_i - p_i)$, and (C) $C^+ = \emptyset$.

Since $\sum_{i \in C^+} (p'_i - p_i) \leq 0$, (b) implies that $\sum_{b \in I^-} (p_b - p'_b) \leq \sum_{i \in B^+} (p'_i - p_i)$. Combined with (A) and (B) it follows that $\sum_{b \in I^-} (p_b - p'_b) = \sum_{i \in B^+} (p'_i - p_i)$ and that all the inequalities in (A) and (B) are actually equations, specifically (c) $\sum_{b \in I^-} (-q_b v_b) = \sum_{i \in B^+} q_i v_i$. Furthermore, (A) implies that the total change in utility (comparing (X, p) to (X', p')) for all bidders $b \in I^-$, which is $\sum_{b \in I^-} (q_b v_b - p'_b + p_b)$, equals 0, and (B) implies that the change in utility for all bidders $i \in B^+$, which is $\sum_{i \in B^+} (q_i v_i - p'_i + p_i)$, equals 0. Since $C^+ = \emptyset$, this implies that the total change in utility for all bidders is zero. The utility of the auctioneer in (X, p) and in (X', p') does not change either. This gives a contradiction to the assumption that (X', p') is Pareto superior to (X, p) and completes the proof of Theorem 3.2.

To show (B) note that the increase in payment $p'_i - p_i$ for a bidder $i \in B$ with $q_i > 0$ is at most $q_i v_i$, otherwise the utility of the bidder would drop. This shows (B). To show the first inequality in (A) note that the total drop in payments by a bidder $b \in I$ with $q_b \leq 0$ is at least $-q_b v_b$. Thus, $\sum_{b \in I^-} (p_b - p'_b) \geq \sum_{b \in I^-} (-q_b v_b)$.

To show the second inequality in (A) we first show the following claims. Let $s = |B^+|$ and let $r(1), r(2), \ldots, r(s)$ be an ordering of the bidders in B^+ in increasing order of $l(\cdot)$ such that two bidders i and i' with $l(i) = l(i')$ are ordered by increasing v-value. We show first that r-ordering orders the bidders by valuation.

CLAIM C.1. *For $1 \leq l < s$ it holds that $v_{r(l)} \leq v_{r(l+1)}$.*

PROOF. Assume by contradiction that $v_{r(l)} > v_{r(l+1)}$ for some $1 \leq l < s$. Since $l(r(l)) < l(r(l+1)) \leq h(r(l+1))$, $r(l+1) \in \tilde{N}_{r(l)}$. Since $r(l) \in B$, it follows that $v_{r(l)} \leq v_{r(l+1)}$. Contradiction! □

Note that $\tilde{N}_{r(l+1)} \subseteq \tilde{N}_{r(l)} \cup \{r(l)\}$ for $1 \leq l < s$, i.e., a bidder $b \in I$ can belong to multiple such sets. We define for each bidder $b \in I^-$ a unique "top" $i \in B^+$ to whose set $\tilde{N}_i$ bidder b belongs. More formally, we define a mapping as follows: Let $p(b) := \arg\max_{i \in B^+ : b \in \tilde{N}_i} r(i)$ which is the maximum $i \in B^+$ (in r-order) with

$b \in \tilde{N}_i$. Let $A_i = \sum_{b \in I^- \cap \tilde{N}_i : p(b) = i} (-q_b)$. By the definition of the mapping p we have that (d) $\sum_{b \in I^- \cap \tilde{N}_{r(1)}} (-q_b v_b) \geq \sum_{b \in I^- \cap \tilde{N}_{r(1)}} (-q_b v_{p(b)}) = \sum_{i \in B^+} (\sum_{b \in I^- \cap \tilde{N}_i : p(b) = i} (-q_b)) v_i = \sum_{i \in B^+} A_i v_i$.

The following claim simply states that all bidders from $r(l)$ to $r(s)$ "receive" all their increases in weighted capacity from bidders in $\tilde{N}_{r(l)}$.

CLAIM C.2. *For all* $1 \leq l \leq s$ *it holds that* $\sum_{l \leq t \leq s} A_{r(t)} = \sum_{b \in I^- \cap \tilde{N}_{r(l)}} (-q_b) \geq \sum_{l \leq t \leq s} q_{r(t)}$.

PROOF. Consider bidders $\{r(t) | t \in \{l, \ldots, s\}\} \subseteq B^+$. We show that bidder i can increase his weighted capacity in the Pareto superior assignment X' only at expenses of the reduction of the weighted capacity of bidders in $\tilde{N}_i$. This in turn implies $\sum_{b \in I^- \cap \tilde{N}_{r(l)}} (-q_b) \geq \sum_{l \leq t \leq s} q_{r(t)}$.

Let us describe the assignment X and the Pareto superior assignment X' by a weighted bipartite directed graph $G = (V, E \cup E')$ with the vertex set $V = I \cup J$, the edge sets $E = \{(i,j) \in I \times J | x_{i,j} > 0\}$ and $E' = \{(j,i) \in J \times I | x'_{i,j} > 0\}$, and the weights $w_{i,j} = x_{i,j}\ \forall (i,j) \in I \times J$ and $w_{j,i} = x'_{i,j}\ \forall (j,i) \in J \times I$. Edges from I to J are weighted by the corresponding real-numbered value $x_{i,j}$. Edges from J to I are weighted by the corresponding real-numbered value $x'_{i,j}$. Consider a path $\pi = (i_1, j_1, i_2, j_2, \ldots, i_{k-1}, j_{k-1}, i_k)$ in the bipartite graph. We say that the path π is an alternating path of length k with respect to the assignments X and X' if $(i_t, j_t) \in E$ and $(j_t, i_{t+1}) \in E'$ for all $1 \leq t < k$. It is an alternating cycle if $i_1 = i_k$. Since for any assignment $\sum_{i \in I} x_{i,j} = 1\ \forall j \in J$, and $\sum_{j \in J} x_{i,j} = \kappa_i\ \forall i \in I$, it holds that

$$\sum_{j \in J} (w_{i,j} - w_{j,i}) = 0\ \forall i \in I, \text{ and} \tag{1}$$

$$\sum_{i \in I} (w_{i,j} - w_{j,i}) = 0\ \forall j \in J. \tag{2}$$

We decompose the bipartite graph in a set of at most $|I|\,|J|$ alternating cycles that we denote by Π. We start from the edge (i,j) or (j,i) with the lowest weight $\lambda = \min_{(x,y) \in E \cup E'} w_{x,y}$. We traverse the bipartite graph starting from edge (x,y) and find a path going from vertex y to vertex x. This gives us a cycle π. If such a path would not exist we could partition the set of vertices into three disjoint subsets: V_1 contains x and all the start vertices of paths ending at x, V_2 contains y and all the end vertices of a paths starting at y, and V_3 contains all the remaining vertices. The edge (x,y) would be directed from a vertex in V_1 to a vertex that is not in V_1 and has a positive weight and no edge would be directed from a vertex that is not in V_1 to a vertex in V_1. Thus, $\sum_{u \in V_1, v \in V_2 \cup V_3} w_{u,v} > 0$ and $\sum_{u \in V_1, v \in V_2 \cup V_3} w_{v,u} = 0$, which would contradict (1) and (2), and hence, a cycle π has to exist.

Let us denote by $\lambda_\pi = \lambda$ the capacity of cycle π. We then reduce by λ_π the weight of all edges on π and we remove from the bipartite graph all edges with 0 remaining weight. Observe that equations (1) and (2) still hold for the resulting graph. It is therefore possible to continue this procedure until the graph is empty.

Given a cycle $\pi = (i_1, j_1, i_2, j_2, \ldots, i_{k-1}, j_{k-1}, i_k)$, we abuse notation by denoting by π also the set of bidders $\{i_1, i_2, \ldots, i_k\}$. For a bidder $i \in \pi$, let us define $t_\pi(i)$ and $t'_\pi(i)$ such that $(i, t_\pi(i)) \in E$ and $(t'_\pi(i), i) \in E'$ are edges of the cycle. We use $\alpha(j)$ for α_j, which is the quality of slot $j \in J$.

Given a bidder $i \in I$ and a set of alternating cycles $\Pi' \subseteq \Pi$ we define

$$q_i(\Pi') = \sum_{\pi \in \Pi' : i \in \pi} \lambda_\pi(\alpha(t'_\pi(i)) - \alpha(t_\pi(i)))$$

as the increase of the weighted capacity of bidder i when moving from the assignment X to a new assignment by the set of cycles Π'. Note that $q_i = q_i(\Pi)$ for every bidder i. It holds for each $\pi \in \Pi$ that

$$\sum_{i \in \pi} q_i(\{\pi\}) = 0. \tag{3}$$

We prove the claim now by induction on a set of cycles Π. We actually prove the stronger statement

$$\sum_{b \in \tilde{N}_{r(l)} \setminus B^+ : q_b(\Pi) \leq 0} (-q_b(\Pi)) \geq \sum_{i \in \tilde{N}_{r(l)} \setminus B^+ : q_i(\Pi) > 0} q_i(\Pi) + \sum_{l \leq t \leq s} q_{r(t)}(\Pi).$$

Observe that the statement above can greatly be simplified by observing that all bidders in $\tilde{N}_{r(l)}$ appear in the above inequality. It is therefore enough to prove for each set of cycles Π that

$$\sum_{i \in \tilde{N}_{r(l)}} q_i(\Pi) \leq 0. \tag{4}$$

It clearly holds for $\Pi = \emptyset$. Assume it holds for Π, we prove in the following that it then holds also for $\Pi' = \Pi \cup \{\pi\}$.

Since

$$\sum_{i \in \tilde{N}_{r(l)}} q_i(\Pi') = \sum_{i \in \tilde{N}_{r(l)}} q_i(\Pi) + \sum_{i \in \tilde{N}_{r(l)} \cap \pi} q_i(\{\pi\}), \tag{5}$$

it is sufficient to prove

$$\sum_{i \in \tilde{N}_{r(l)} \cap \pi} q_i(\{\pi\}) \leq 0. \tag{6}$$

For any bidder $s \in \tilde{N}_{r(l)}$ and for any bidder $i \notin \tilde{N}_{r(l)}$ it holds that $h(i) \leq l(s)$. This implies in turn that any bidder in $i \in \pi \cap \tilde{N}_{r(l)}$ will only increase his weighted capacity when swapping a fraction of a slot against a fraction of a slot that is assigned to another bidder $s \in \pi \cap \tilde{N}_{r(l)}$ in X. It follows $\sum_{i \in \pi \setminus \tilde{N}_{r(l)}} q_i(\{\pi\}) \geq 0$. Combined with Equation (3) this yields the proof of the statement of Equation (6). □

We need one more auxiliary claim before completing the proof of the second inequality of (A).

CLAIM C.3. *If (X', p') is a Pareto superior solution to (X, p) then for every $1 \leq l \leq s$*

$$\sum_{l \leq t \leq s} A_{r(t)} v_{r(t)} \geq \sum_{l \leq t \leq s} q_{r(t)} v_{r(t)} + \sum_{l \leq t \leq s} (A_{r(t)} - q_{r(t)}) v_{r(l)}.$$

PROOF. We use backwards induction on l. For $l = s$, it trivially holds that $A_{r(s)} v_{r(s)} \geq q_{r(s)} v_{r(s)} + (A_{r(s)} - q_{r(s)}) v_{r(s)}$.

For $l < s$, we use the inductive claim for $l + 1$, Claim C.2, and the fact that $v_{r(l+1)} \geq v_{r(l)}$ according to Claim C.1. Thus, $\sum_{l \leq t \leq s} A_{r(t)} v_{r(t)} = \sum_{l+1 \leq t \leq s} A_{r(t)} v_{r(t)} + A_{r(l)} v_{r(l)} \geq \sum_{l+1 \leq t \leq s} q_{r(t)} v_{r(t)} +$

$\sum_{l+1 \leq t \leq s}(A_{r(t)} - q_{r(t)})v_{r(l+1)} + A_{r(l)}v_{r(l)} \geq \sum_{l \leq t \leq s} q_{r(t)}v_{r(t)} + \sum_{l+1 \leq t \leq s}(A_{r(t)} - q_{r(t)})v_{r(l)} + (A_{r(l)} - q_{r(l)})v_{r(l)} = \sum_{l \leq t \leq s} q_{r(t)}v_{r(t)} + \sum_{l \leq t \leq s}(A_{r(t)} - q_{r(t)})v_{r(l)}$.

□

By Claim C.2 it follows that $\sum_{1 \leq t \leq s}(A_{r(t)} - q_{r(t)}) \geq 0$, and thus, by (d) and Claim C.3 it follows that $\sum_{b \in I^-}(-q_b)v_b \geq \sum_{i \in B^+} A_i v_i = \sum_{1 \leq t \leq s} A_{r(t)}v_{r(t)} \geq \sum_{1 \leq t \leq s} q_{r(t)}v_{r(t)} = \sum_{i \in B^+} q_i v_i$. This completes the proof of the second inequality of (A).

To show (C) assume by contradiction that $C^+ \neq \emptyset$ and consider two cases that follow from Claim C.2:

Case 1: $\sum_{i \in B^+} A_i > \sum_{i \in B^+} q_i$. Combined with (d) and Claim C.3 this shows that $\sum_{b \in I^-}(-q_b v_b) \geq \sum_{i \in B^+} A_i v_i > \sum_{i \in B^+} q_i v_i$. But this is a contradiction to (c) above.

Case 2: $\sum_{i \in B^+} A_i = \sum_{i \in B^+} q_i$. Note that $\sum_{i \in B^+} A_i = \sum_{b \in I^- \cap \tilde{N}_{r(1)}}(-q_b)$. Then (a) implies $\sum_{b \in I^- \setminus \tilde{N}_{r(1)}}(-q_b) = \sum_{i \in C^+} q_i > 0$. By (c) $\sum_{i \in B^+} q_i v_i = \sum_{b \in I^-}(-q_b v_b) = \sum_{b \in I^- \cap \tilde{N}_{r(1)}}(-q_b v_b) + \sum_{b \in I^- \setminus \tilde{N}_{r(1)}}(-q_b v_b) > \sum_{b \in I^- \cap \tilde{N}_{r(1)}}(-q_b v_b)$. By Claim C.2, Claim C.3, and (d) it follows that $\sum_{b \in I^- \cap \tilde{N}_{r(1)}}(-q_b v_b) \geq \sum_{i \in B^+} A_i v_i \geq \sum_{i \in B^+} q_i v_i$, which contradicts the previous statement.

D. EXAMPLE FOR THE DIVISIBLE CASE

Bidder 1 has valuation $v_1 = 1$, budget $b_1 = 1$, and slot constraint $\kappa_1 = 1$. Bidder 2 has valuation $v_2 = 2$, budget $b_2 = 0.5$, and slot constraint $\kappa_2 = 1$. The qualities of the slots are $\alpha_1 = 1$ and $\alpha_2 = 2$. The auction starts for both bidders with a price of zero and thus their demand is infinite. First we call SELL for bidder 1. He gets a weighted capacity of one for price zero, since the most weighted capacity that we can assign to bidder 2 is the quality of slot 2. Then we set the price of bidder 1 to one and call SELL for bidder 2. After this call we sell a weighted capacity of one to bidder 2, since the most weighted capacity that we can assign to bidder 1 is the quality of slot 2 and he can also afford just an additional weighted capacity of one. Then we set the price of bidder 2 to one and continue with the next iteration. Bidder 1 becomes an exiting bidder and we call SELL for him. Bidder 2 can only afford an additional weighted capacity of one half. Hence, we have to sell the other half that is left to bidder 1. Next we sell the other half to bidder 2. Each bidder gets a weighted capacity of one and a half and pays a half. The only possible assignment is that each bidder gets half of the first slot and half of the second slot.

E. EXISTENCE OF FEASIBLE SOLUTIONS IN ALGORITHM SELL

LEMMA E.1. *For every execution of procedure* SELL *there exists a feasible solution to the linear program in the call.*

PROOF. We show the claim by induction on the number t of calls to procedure SELL. There is a feasible solution for the first call to SELL as the demand of every bidder is unlimited and, thus, γ_i can be made as large as necessary for every bidder i. Next let us inductively assume that there was a feasible solution for call t and let us consider call $t+1$. As there exists a feasible solution for call t, SELL returns an optimal solution (X, γ) for call t. After the call, $c_{i'}$ is increased by $\gamma_{i'}$, and thus, $(X, \tilde{\gamma})$ with $\tilde{\gamma}_i = \gamma_i$ for $i \neq i'$ and $\tilde{\gamma}_{i'} = 0$ for $i = i'$ is a feasible solution of the linear program in call $t+1$ to SELL, which uses the new c-values. Since $\tilde{\gamma}_{i'} = 0$, $(X, \tilde{\gamma})$ is a feasible solution in call $t+1$ even if the price for bidder i' was increased, and thus, his demand $d_{i'}$ was decreased. Thus the inductive claim holds. □

F. PROOF OF PROPOSITION 3.3

Since no bidder will ever pay more than his reported valuation and the demand is set so that $b_i \geq p_i$, individual rationality follows.

We next show incentive compatibility. By the construction of the auction, each bidder i never pays more than his reported valuation. If his reported valuation is $\tilde{v}_i$ and $\tilde{v}_i < v_i$, he becomes an inactive bidder at the price $\tilde{v}_i$. His utility cannot increase by lying as he gets the same weighted capacity for each price $\pi < \tilde{v}_i$ and he will loose all weighted capacity that he clinched at a price larger than $\tilde{v}_i$. If his reported valuation is $\tilde{v}_i > v_i$, he gets the same weighted capacity for each price $\pi < v_i$. He might receive additional weighted capacity at a price at least v_i, but it cannot increase his utility. Thus, the auction is IC.

All the coefficients of the affine functions used in the constraints of the first linear program that gets solved during the auction are rational numbers and all the linear programs have feasible solutions. Thus, there exists an optimal solution that is a vertex of the polytope that is defined by the constraints of the respective linear program. Since that optimal solution lies on the intersection of the graphs of affine functions with rational coefficients it follows that the selected optimal solution (X, γ) has only rational entries. The prices are rational numbers as well, and thus, c_i and d_i are rational numbers for all $i \in I$ in the next iteration. Hence, the allocation (X, p) that is determined by the auction has only rational entries.

G. PROOF OF THEOREM 3.4

We will show that (X, p) does not contain any trading swap. Let (X, p) be the allocation computed by the auction and assume by contradiction that there exists a trading swap, i.e., a sequence of bidders $(u = a_0, a_1, \ldots, a_k = w)$ that fulfills the above conditions. Consider the Pareto superior allocation (X', p') constructed in the proof of Theorem 3.1. Define $c_i^{\mathrm{f}} := \sum_{j \in J} \alpha_j x_{i,j}$ and $c_i' := \sum_{j \in J} \alpha_j x_{i,j}'$ for all bidders i. Note that $c_w' = c_w^{\mathrm{f}} - \delta$, $c_u' = c_u^{\mathrm{f}} + \delta$, and $c_i^{\mathrm{f}} = c_i'\ \forall i \in I \setminus \{u, w\}$. Let $\delta' = \delta \frac{\pi}{\pi^+} > 0$.

We construct a modified Pareto superior allocation (X'', p'') with $c_w'' = c_w^{\mathrm{f}} - \delta'$, $c_u'' = c_u^{\mathrm{f}} + \delta'$, and $c_i'' = c_i^{\mathrm{f}}\ \forall i \in I \setminus \{u, w\}$, where $c_i'' = \sum_{j \in J} \alpha_j x_{i,j}''$. Specifically, we use the same set of bidders $u = a_0, \ldots, a_k = w$, perform the swaps between the same bidders as for (X', p'), but use as swap values $\tau_{p+1}' := \tau_{p+1} \frac{\pi}{\pi^+}$ instead of τ_{p+1} and as payments $p_u'' = p_u + v_w \delta'$, $p_w'' = p_w - v_w \delta'$, and $p_i'' = p_i$ for all other bidders i. By the same argument as for (X', p') the allocation (X'', p'') is Pareto superior to (X, p).

We will show that (X'', p'') can be used to construct a smaller feasible solution to one of the linear programs solved by SELL. Since the linear program has found the minimal solution this leads to a contradiction with the assumption that there exists a trading swap in (X, p).

Let $b_i^{\mathrm{f}} := b_i - p_i$ be the remaining budget of bidder i at the end of the algorithm. The value c_w of bidder w increases only when procedure SELL returns a non-zero value for γ_w, where w was the last parameter when SELL was called, that is, the linear program solved in SELL was trying to minimize γ_w. Since $c_w^{\mathrm{f}} > c_w''$, there exists a unique call to procedure SELL with parameters $(I, J, \alpha, \kappa, v, c, d, w)$ such that *before* the execution of the linear program $c_w \leq c_w''$ and SELL returns a value $s > 0$ such that $c_w + s > c_w''$. We call the corresponding linear program LP. Its inputs are the vectors c, d, and κ, its variables are the matrix $X = (x_{i,j})_{(i,j) \in I \times J}$ and the vector γ. Let π be the price at the time of the call. We will show that using (X'', p'') we can construct a feasible solution for this linear program which outputs a value $s' < s$. This leads to the desired contradiction. We first show the following claim:

CLAIM G.1. *Using (X,p) we can find a feasible solution $(X,\tilde{\gamma})$ to LP with $\tilde{\gamma}_i = \sum_{j\in J} x_{i,j}\alpha_j - c_i \; \forall i \in I$ that fulfills (1) for all bidders $i \in A \setminus E$ with $i \neq w$ and $d_i > d_i^+$: $\tilde{\gamma}_i \leq d_i - \frac{b_i^{\mathrm{f}}}{\pi}$, and (2) for all bidders $i \in A \setminus E$ with $i \neq w$ and $d_i = d_i^+$: $\tilde{\gamma}_i \leq d_i - \frac{b_i^{\mathrm{f}}}{\pi^+}$.*

PROOF. We first recall that the case $d_i > d_i^+$ happens when the LP is computed before the demand of bidder i has been updated as in line 16 of AUCTION. The case $d_i = d_i^+$ happens after the update has been made for bidder i.

First we show that $(X,\tilde{\gamma})$ fulfills the constraints of LP. Since the allocation (X,p) is derived from the last linear program executed by the algorithm, it fulfills the conditions $\sum_{i\in I} x_{i,j} = 1 \; \forall j \in J$ and $\sum_{j\in J} x_{i,j} = \kappa_i \; \forall i \in I$. By definition $\sum_{j\in J} x_{i,j}\alpha_j - \tilde{\gamma}_i = c_i \; \forall i \in I$. Recall that b_i^{f} is the remaining budget of bidder i at the end of the auction, that is, the money *not* spent by i. Note that bidder i clinched $\tilde{\gamma}_i = c_i^{\mathrm{f}} - c_i$ "weighted capacity" after LP was executed.

Case 1: Consider first a bidder $i \neq w$ with $d_i > d_i^+$. Note that for bidders of this type the remaining budget when LP is called is $d_i\pi$ and that these bidders pay a price per "weighted capacity unit" of at least π for all capacity that was not clinched before LP was executed. Thus, bidder i pays $d_i\pi - b_i^{\mathrm{f}}$ for all the "weighted capacity" that was not clinched before LP was executed. Thus, $\tilde{\gamma}_i\pi \leq d_i\pi - b_i^{\mathrm{f}}$.

Case 2: Consider next a bidder $i \neq w$ with $d_i = d_i^+$. Note that for bidders of this type the remaining budget when LP is called is $d_i\pi^+$ and that these bidders pay a price per "weighted capacity unit" of at least π^+ for all capacity that was not clinched before LP since they can only clinch at the price π^+ or higher. Note that we know that $\pi^+ \leq v_i$: Since $i \in A \setminus E$ it holds that $v_i > \pi$, and therefore, $v_i \geq \pi^+$. Thus, bidder i pays $d_i\pi^+ - b_i^{\mathrm{f}}$ for all the "weighted capacity" clinched after LP was executed. Thus, $\tilde{\gamma}_i\pi^+ \leq d_i\pi^+ - b_i^{\mathrm{f}}$. □

Next we define $\gamma_i'' = \sum_{j\in J} x_{i,j}''\alpha_j - c_i = c_i'' - c_i$ for all $i \in I$ and show that (X'', γ'') is a feasible solution of LP and that $\gamma_w'' < s$ thus leading to a contradiction. Note that $\gamma_u'' = \tilde{\gamma}_u + \delta'$. By the definition of X'' for all $i \in I$ it holds that $\sum_{j\in J} x_{i,j}'' = \sum_{j\in J} x_{i,j} = \kappa_i$ as whenever for some τ with $-1 \leq \tau \leq 1$, $x_{i,j}''$ is set to $x_{i,j} + \tau$ for some $j \in J$, $x_{i,l}''$ is set to $x_{i,l} - \tau$ for some other $l \in J$. Additionally for every $j \in J$ it holds that $\sum_{i\in I} x_{i,j}'' = \sum_{i\in I} x_{i,j} = 1$ as whenever $x_{a_p,j}''$ is set to $x_{a_p,j} + \tau$ for some τ with $-1 \leq \tau \leq 1$, either $x_{a_{p+1},j}''$ is set to $x_{a_{p+1},j} - \tau$, or $x_{a_{p-1},j}''$ is set to $x_{a_{p-1},j} - \tau$. Thus (X'', γ'') fulfills constraints (a) and (b) of LP. By the definition of γ'' constraint (c) also holds.

For constraint (d) note that for all $i \in I \setminus \{u, w\}$ we know that $\gamma_i'' = \tilde{\gamma}_i \leq d_i$, and thus, constraint (d) holds for such i. For $i = w$, by definition of a trading swap $\sum_{j\in J} \alpha_j x_{i,j}'' < \sum_{j\in J} \alpha_j x_{i,j}$, and thus, $\gamma_w'' < \tilde{\gamma}_w \leq d_w$. Hence constraint (d) also holds for $i = w$. For $i = u$, we know that $\gamma_u'' = \tilde{\gamma}_u + \delta'$ and we have to show that $d_u \geq \gamma_u''$.

Since $c_w^{\mathrm{f}} > c_w$ we know that w is still an active bidder when LP is executed, and thus, $v_w \geq \pi$. Hence, $b_u^{\mathrm{f}} = b_u - p_u \geq p_u' - p_u = v_w\delta \geq \pi\delta$.

By $v_w \geq \pi$ it follows from the definition of a trading swap that $v_u > \pi$ and that therefore $u \in A \setminus E$. Consider first the case that $d_u > d_u^+$. By the previous claim it follows that $d_u \geq \tilde{\gamma}_u + \frac{b_u^{\mathrm{f}}}{\pi} \geq \tilde{\gamma}_u + \delta = \gamma_u'' + \delta - \delta' > \gamma_u''$. Consider next the case that $d_u = d_u^+$. By the previous claim it follows that $d_u \geq \tilde{\gamma}_u + \frac{b_u^{\mathrm{f}}}{\pi^+} \geq \tilde{\gamma}_u + \delta\frac{\pi}{\pi^+} = \tilde{\gamma}_u + \delta' = \gamma_u''$.

It remains to show that $\gamma_w'' < s$. Recall that by the definition of LP it holds that $c_w + s > c_w''$, while, by definition of γ_w'', $c_w + \gamma_w'' = c_w''$. Thus $\gamma_w'' < s$, which leads to the desired contradiction.

H. PROOF OF LEMMA 4.1

We first show the following claim:

CLAIM H.1. *For every feasible allocation (N, p) in the indivisible case there exists a feasible allocation (X, p) in the divisible case where all the bidders and the auctioneer have the same utility.*

PROOF. The utility of the auctioneer stays unchanged, since we leave the payments unchanged. We set $x_{i,j} = \frac{|\{r \in R | n_{j,r} = i\}|}{|R|} \; \forall i \in I, \; \forall j \in J$. The utility of bidder i is the same for (N, p) and (X, p), since the utility of bidder i is $\sum_{j \in J} \frac{\alpha_j}{|R|} |\{r \in R | n_{j,r} = i\}| v_i - p_i = \sum_{j \in J} \alpha_j x_{i,j} v_i - p_i$ for (N, p). The slot constraint for (N, p) implies $\kappa_i \geq \max_{r \in R} |\{j \in J | n_{j,r} = i\}| \geq \frac{|\{(j,r) \in J \times R | n_{j,r} = i\}|}{|R|} = \sum_{j \in J} \frac{|\{r \in R | n_{j,r} = i\}|}{|R|} = \sum_{j \in J} x_{i,j}$, and therefore it implies the slot constraint in (X, p). Since all the slots are fully assigned to the bidders in (N, p), and consequently for (X, p), it follows that (X, p) is feasible. □

Given a probability distribution over feasible allocations for the indivisible case, transform each feasible allocation that has a non-zero probability into a feasible allocation for the divisible case. Then create a new allocation for the divisible case by adding up the all of these feasible allocations for the divisible case weighted by the probability distribution. Since the weights are created by a probability distribution, they add up to 1, and thus, the resulting combined allocation fulfills Conditions (1) and (2) of a feasible allocation. As the payment is identical to the payment for the indivisible case, Condition (3) is also fulfilled.

I. CONSTRUCTION OF MATRIX M'

We describe next how to construct M'. Recall that all $x^d_{i,j}$ are rational numbers. Let λ be their least common denominator, set $C = \{1, \ldots, \lambda\}$, and set $y_{i,j} = \lambda x^d_{i,j}$. Since $\sum_{i \in I} x^d_{i,j} = 1$ and $\sum_{j \in J} x^d_{i,j} \leq \kappa_i$, we know that $\sum_{i \in I} y_{i,j} = \lambda$ and $\sum_{j \in J} y_{i,j} \leq \lambda \kappa_i$. We construct a matrix M of size $|J| \times \lambda$ with values in I by setting $y_{i,j}$ values of row j to i. More formally, for each $j \in J$ and each $c \in C$ we set entry $m_{j,c} = v$ for the unique value v with $\sum_{i=1}^{v-1} y_{i,j} < c$ and $\sum_{i=1}^{v} y_{i,j} \geq c$. As a result $|\{c \in C | m_{j,c} = i\}| = y_{i,j} \; \forall i \in I, \forall j \in J$. The slot constraints imply that there are at most $\lambda \kappa_i$ entries in M that have the value $i \in I$. Next, we replace each bidder i by κ_i *pseudo-bidders* and translate M into a matrix P such that no pseudo-bidder has more than λ entries in P.

We construct P in the following way: we give every entry $(j, c) \in J \times C$ of M that has the value $i \in I$ a unique number $l_{j,c}$ that is starting from 1; we assign the same value to two entries (j, c) and (j', c') in P if and only if the corresponding entries in M have the same values ($m_{j,c} = m_{j',c'}$) and their numbers $l_{j,c}$ and $l_{j',c'}$ fulfill $\lfloor \frac{l_{j,c}}{\lambda} \rfloor = \lfloor \frac{l_{j',c'}}{\lambda} \rfloor$. To be more concrete, let the indicator variable $\delta_{i,j,c} = 1$ if $m_{j,c} = i$ and $\delta_{i,j,c} = 0$ otherwise. We define $l_{j,c} := \sum_{(j',c') \in S_{j,c}} \delta_{m_{j,c},j',c'}$ where $S_{j,c} = \{(j', c') \in J \times C | (j' < j) \vee (j' = j \wedge c' \leq c)\}$ and construct matrix P of size $|J| \times \lambda$ by $p_{j,c} = \left(\sum_{i'=1}^{m_{j,c}-1} \kappa_{i'} \right) + \lfloor \frac{l_{j,c}}{\lambda} \rfloor + 1$. The matrix P has the property that all the entries that have an identical value in P have an identical value in M, but every value appears at most λ times in P.

Then we apply to P a swapping algorithm that gives us P' and that guarantees that (1) in each column in P there is at most one entry for each pseudo-bidder, and (2) for each $j \in J$ each value appears as often in row j of P as it does in P'. Thus, when we convert all the entries of the pseudo-bidders of a given bidder i into entries for bidder i we get a matrix M' such that each bidder i has at most κ_i entries in each column and for all $i \in I$ and $j \in J$ it holds that $|\{c \in C | m'_{j,c} = i\}| = y_{i,j}$. The swapping algorithm that turns P into P' is presented in Algorithm 3.

Algorithm 3 Swapping Algorithm.

procedure SWAPPING(M)
    $J \leftarrow \{1, \ldots, \text{rows}(M)\}$
    $C \leftarrow \{1, \ldots, \text{columns}(M)\}$
    $n \leftarrow \max_{(j,c) \in J \times C} m_{j,c}$
    for $i \in \{1, \ldots, n\}$ **do**
        $j_c \leftarrow |\{j \in J | m_{j,c} = i\}| \; \forall c \in C$
        while $\max_{c \in C}(j_c) > 1$ **do**
            $a \leftarrow \min(\{c \in C | j_c > 1\})$
            $b \leftarrow \min(\{c \in C | j_c = 0\})$
            $i' \leftarrow i$
            $k \leftarrow 0$
            repeat
                $k \leftarrow \min(\{j \in J \setminus \{k\} | m_{j,a} = i'\})$
                $m_{k,a} \leftarrow m_{k,b}$
                $m_{k,b} \leftarrow i'$
                $i' \leftarrow m_{k,a}$
            until $|\{j \in J | m_{j,a} = i'\}| = 1 \vee |\{j \in J | m_{j,b} = i'\}| > 0$
            $j_a \leftarrow j_a - 1$
            $j_b \leftarrow j_b + 1$
        end while
    end for
    return M
end procedure

THEOREM I.1. *Given a matrix M of size $r \times n$ with entries valued in I and where each value appears in at most n entries, there exists a swapping algorithm that finds a matrix M' with the same size and where (1) each value appears as often in row j of M' as it appears in row j of M and (2) each value appears in at most one entry of each column of M'.*

PROOF. Our goal is to find an algorithm that swaps the values between the entries such that each value appears only once in each column. We define $J = \{1, \ldots, r\}$ and $C = \{1, \ldots, n\}$. Let the indicator variable $\delta_{i,j,c} = 1$ if $m_{j,c} = i$, $\delta_{i,j,c} = 0$ otherwise. We define the badness of a value $i \in I$ as $\beta_i(M) = \sum_{j \in J} \sum_{c \in C} (\delta_{i,j,c}) - |\{c \in C | \sum_{j \in J} \delta_{i,j,c} > 0\}|$ as the difference between the number of entries which have value i and the number of columns in which i appears. Moreover, we define by $\beta(M) = \sum_{i \in I} \beta_i(M)$ the badness of matrix M. When each value appears at most once in each column, the badness of the matrix is 0. We aim at reducing the badness of the matrix at each sequence of swaps.

Let us assume that i appears more than once in column a. Then, there exists a column b where i does not appear at all, because each value appears in at most n entries. For the following operations we consider only the columns a and b. We now define a sequence of swaps between pairs of entries of the two columns. We can see the two columns as the two sides of a bipartite graph. We set vertices $A = \{a_1, \ldots, a_r\}$ on the left side and vertices $B = \{b_1, \ldots, b_r\}$ on the right side. The values of the a_j and b_j are $m_{j,a}$ and $m_{j,b}$, respectively for all $j \in J$. We set edges $\{(a_j, b_j) | j = 1, \ldots, r\}$ from left vertices to right vertices of the same slot, and edges $\{(b_j, a_k) | m_{j,b} = m_{k,a}\}$ from right vertices to left vertices with same value.

We define a swapping alternating path $(a_{j_1}, b_{j_1}, \ldots, a_{j_t}, b_{j_t})$ on the bipartite graph. The path starts with a vertex of the left side and ends with a vertex of the right side. We start with a_{j_1}, one of the vertices on the left side with value i, and set $i_0 = i$. Vertex a_{j_k} is followed in the path by vertex b_{j_k}. Let $m_{j_k,b} = i_k$ be the value of vertex b_{j_k}. If value i_k appears more than once on column b then we end the path. Otherwise, we continue to the left with any of the edges $(b_{j_k}, a_{j_{k+1}})$, if any such edge exists. Finally, we implement t swaps by exchanging the values of the endpoints (a_{j_k}, b_{j_k}) of each edge on the path, i.e., we exchange $m_{j_k,a}$ with $m_{j_k,b}$.

We prove two claims:

CLAIM I.2. *The sequence $(i_1, \ldots, i_t)$ does not contain any value more than once and it does not contain value i_0.*

PROOF. Assume that the path has reached vertex b_{j_k} on the right side of the graph. The path continues to the left side only if the value i_k appears only once on the right side. Therefore, the sequence $(i_1, \ldots, i_t)$ contains the values of the vertices on the right side only once. The set does not contain $i_0 = i$ since value i does not appear on b. □

CLAIM I.3. *The sequence of swaps along the edges (a_{j_k}, b_{j_k}), $k = 1, \ldots, t$, reduces the total badness of the matrix by at least 1.*

PROOF. The first swap of the path reduces the badness of bidder $i = i_0$ by 1, since there exists no value i on any entry of column b. We now prove that the total badness of bidders $\{i_1, \ldots, i_t\}$ does not increase. Consider any value i_k with $k < t$. If value i_k is moved to entry $m_{j_k,a}$ then i_k appears only once in b. However, value i_k is also moved from entry $m_{j_{k+1},a}$ to $m_{j_{k+1},b}$. Thus, the badness of bidder i_k does not increase. For value i_t, which is moved from $m_{j_t,b}$ to $m_{j_t,a}$, we observe that either value i_t appears more than once in b or that there is no entry on a that contains i_t. In both cases the badness of i_t does not increase. □

We can now use the swapping algorithm on M and get a matrix M' of size $r \times n$ where none value appears more than once in the same column. The proof of the theorem is therefore completed. □

To construct matrix M' we simply reproduce the swaps that happened to matrix P on matrix M. We define the matrix M' with size $|J| \times \lambda$ where $m'_{j,c} = \min\{v \in I | \sum_{i=1}^{v-1} \kappa_i < m_{j,c}\}$. The values of the entries of M' correspond to the bidders in I and in each column each value $i \in I$ appears at most κ_i times. This methodology preserves the amount of capacity of each slot $j \in J$ that is allocated to each bidder $i \in I$ (i.e., $|\{c \in C | m'_{j,c} = i\}| = \sum_{c \in C} \delta_{i,j,c} = y_{i,j}\ \forall i \in I, \forall j \in J$). The columns of matrix M' are then used in the sampling step.

The allocation of the randomized auction for multiple keywords and indivisible slots is now constructed as follows. We select for each keyword r randomly and with equal probability $\frac{1}{\lambda}$ a column l of matrix M' and set $z_{i,j,r} = 1$ if $m'_{j,l} = i$ and zero otherwise. If $z_{i,j,r} = 1$ slot j of keyword r is assigned to bidder i. The expected weighted capacity allocated to bidder $i \in I$ is thus

$$\mathbf{E}(\sum_{j \in J} \frac{\alpha_j}{|R|} \sum_{r \in R} z_{i,j,r}) = \sum_{j \in J} \frac{\alpha_j}{|R|} \sum_{r \in R} \mathbf{E} z_{i,j,r} = \sum_{j \in J} \alpha_j \frac{y_{i,j}}{\lambda} = \sum_{j \in J} \alpha_j x^d_{i,j}.$$

Additionally, all of the slots are fully assigned to the bidders, and hence, the stated properties are fulfilled by the randomized auction.

J. EXAMPLE FOR THE INDIVISIBLE CASE

We give an example that continues the example given in Appendix D. Recall that each bidder gets half of the first slot and half of the second slot and assume that we need an assignment for three keywords. The least common denominator of the assignments is two, and the matrix M has two columns $(1,1)'$ and $(2,2)'$. Thus, the swapping algorithm returns the matrix M', which has the two columns $(1,2)'$ and $(2,1)'$. We can now select with equal probability one of the two columns of M' for each of the three keywords.

K. THE COMBINATORIAL CASE WITH MULTIPLE SLOTS

We give a *deterministic* mechanism for a special case of the combinatorial setting. The single-valued combinatorial auction of Fiat et al. [2011] solves the budgeted auctioning problem where different bidders are interested in different subsets of the keywords, and each keyword has only *one* slot. Every bidder still values each keyword in her subset the same. Valuations are additive. We extend their techniques to the multi-slot per keyword setting as follows: (1) We extend their B-matchings based approach by giving capacities, equal to the number of unsold slots, to nodes that represent keywords. (2) We extend the concept of trading alternating paths in the bidder/keyword bipartite graph, which in turn allows us to give a characterization of Pareto optimality for the multi-slot case. While in the single-slot case it is sufficient to restrict the attention to simple trading alternating paths, in our case there might be trading options where the same bidder or item can appear many times along the same path. The crucial insight is that there always exists a simple trading path whenever there exists a non-simple one.

We slightly abuse notation by using H to denote both: the sets of the items that are allocated to the bidders, and the B-matching that describes the allocation in graph G.

Pareto optimality has been related in previous work [Dobzinski et al. 2008; 2011] to the non-existence of trading options between bidders. We need a new definition of a trading path because we consider multisets of items.

Definition K.1. A path $\sigma = (a_1, t_1, a_2, t_2, \ldots, a_{j-1}, t_{j-1}, a_j)$ is an alternating path with respect to an assignment H if $(a_i, t_i) \in H$, $t_i \in S_{i+1}$, and $t_i \notin H_{i+1}$ for all $1 \leq i < j$.

Definition K.2. A path $\sigma = (a_1, t_1, a_2, t_2, \ldots, a_{j-1}, t_{j-1}, a_j)$ is a *trading path* with respect to allocation (H, p) if the following holds: (1) σ is an alternating path in H, (2) the valuation of bidder a_j is strictly greater than the valuation of bidder a_1 (i.e., $v_{a_j} > v_{a_1}$), (3) the remaining (unused) budget $b^*_{a_j}$ of bidder a_j at the conclusion of the auction is at least the valuation of bidder a_1 (i.e., $b^*_{a_j} \geq v_{a_1}$).

Observe that the condition $t_i \notin H_{i+1}$ is needed in this case since the slots of a keyword have to be assigned to different bidders. This is not the case in the definition of alternating paths given in [Fiat et al. 2011]. Furthermore all previous results only have to deal with setting where the only possible trading paths are simple alternating paths. We do not require that alternating paths are simple (without cycles) (as in [Fiat et al. 2011]). For example, assume that we have two keywords t_1, t_2 with two slots and five bidders a_1, a_2, a_3, a_4 and a_5 with $S_1 = S_3 = \{t_1\}$ and $S_2 = S_4 = S_5 = \{t_1, t_2\}$. Assume further that the current allocation H is $H_1 = \emptyset$, $H_2 = H_5 = \{t_2\}$, and $H_3 = H_4 = \{t_1\}$. Now $\sigma = (a_3, t_1, a_2, t_3, a_4, t_1, a_1)$ is an alternating path with respect to H. Here, the item t_1 appears twice, i.e., there is a cycle inside the path. Non-simple alternating paths might cause problems as our mechanism can only guarantee that at termination there are no *simple* alternating paths in (H, p) (see Theorem K.4). Let us call two alternating paths *Pareto equivalent* if they have the same start and end bidders and produce the same change in weighted capacity for all the bidders. As the following lemma shows we do not need the restriction to simple alternating paths as whenever there is a non-simple alternating path then there is a Pareto equivalent simple alternating path. In the above example, $\sigma' = (a_3, t_1, a_1)$ would be such a Pareto equivalent simple alternating path.

LEMMA K.3. *If there exists an alternating path $\sigma = (a_1, t_1, \ldots, t_{j-1}, a_j)$ that contains cycles there exists also a Pareto equivalent simple alternating path.*

PROOF. Assume that $\sigma = (a_1, t_1, \ldots, t_{j-1}, a_j)$ is an alternating path with a cycle. Let s_k be the first vertex of the cycle in the order. We decompose the path into $\sigma_s = (a_1, \ldots, s_{k-1}, s_k)$, $\sigma_c = (s_k, s_{k+1}, \ldots, s_{i-1}, s_k)$, and $\sigma_e = (s_k, s_{i+1}, \ldots, a_j)$, where s_k is either a bidder or an item. If s_k is an item, say t, then s_{k-1}, s_{k+1}, s_{i-1}, and s_{i+1} are bidders, and $t \in S_{s_{k-1}}$, $t \in S_{s_{k+1}}$, $t \in S_{s_{i-1}}$, and $t \in S_{s_{i+1}}$. Moreover, $t \in H_{s_{k-1}}$, $t \in H_{s_{i-1}}$, $t \notin H_{s_{k+1}}$, and $t \notin H_{s_{i+1}}$ by the definition of an alternating path. Thus, the concatenation of σ_s and σ_e is still an alternating path and it is simple. In the same way, if s_k is a bidder, say a, we have that s_{k-1}, s_{k+1}, s_{i-1}, and s_{i+1} are items and $\{s_{k-1}, s_{k+1}, s_{i-1}, s_{i+1}\} \subseteq S_a$, $s_{k-1} \notin H_a, s_{i-1} \notin H_a$, $s_{k+1} \in H_a, s_{i+1} \in H_a$. Thus, we can again concatenate σ_s and σ_e and obtain a simple alternating path. The above process can be iterated if there exist more cycles in the path. □

Pareto optimality and simple trading paths are now related by the following theorem.

THEOREM K.4. *Any allocation $(H,\, p)$ is PO if and only if (1) all slots of the keywords are sold in $(H,\, p)$, and (2) there are no simple trading paths in $(H,\, p)$.*

PROOF. In order to prove Theorem K.4 we need the following lemmas and the definition of Pareto equivalent trading paths.

LEMMA K.5. *Let H and $H^{'}$ be two allocations with all items allocated. The symmetric difference $H \ominus H^{'}$ between the two allocations can be decomposed into a set of alternating paths (simple paths and paths with cycles) with respect to H.*

PROOF. Let the graph G be a directed graph. The edges from the matching H are directed from bidders to items and the edges from the matching $H^{'}$ are directed from items to bidders. Since H and $H^{'}$ have all items allocated, in both matchings there are m edges for every item. Moreover, each item in G will have an equal number of incoming and outgoing edges. Thus, no item has to be the start or the end of a path, and we can always find two edges incident to any item such that there are no two consecutive edges from H or $H^{'}$. □

We conclude that for every trading path there is a Pareto equivalent simple trading path and that if there are no simple trading paths then there are no trading paths at all.

Now we are ready for the proof of Theorem K.4. This proof and the proof of Theorem 5.1 follow very closely those of Fiat et al. [2011].

Let $\mathcal{Q}$ be the predicate that (H,p) is PO, $\mathcal{R}_1$ be the predicate that all items are sold in (H,p), and $\mathcal{R}_2$ the predicate that there are no trading paths in G with respect to (H,p). We seek to show that $\mathcal{Q} \Leftrightarrow \mathcal{R}_1 \wedge \mathcal{R}_2$.

$\mathcal{Q} \Rightarrow (\mathcal{R}_1 \wedge \mathcal{R}_2)$: to prove this we show that $(\neg\mathcal{R}_1 \vee \neg\mathcal{R}_2) \Rightarrow \neg\mathcal{Q}$.

If both $\mathcal{R}_1$ and $\mathcal{R}_2$ are true then this becomes $\mathrm{False} \Rightarrow \neg\mathcal{Q}$ which is trivially true.

If the allocation (H,p) does not assign all items ($\neg\mathcal{R}_1$) then it is clearly not PO ($\neg\mathcal{Q}$). We can get a better allocation by assigning unsold items to any bidder i with such items in S_i. This increases the utility of bidder i.

If $\neg\mathcal{R}_2$ then there exists a trading path σ in G with respect to (H,p). Let $\sigma = (a_1, t_1, a_2, t_2, \ldots, a_{j-1}, t_{j-1}, a_j)$, with $v_{a_j} > v_{a_1}$ and $b^*_{a_j} \geq v_{a_1}$; then we can decrease the payment of bidder a_1 by v_{a_1}, increase the payment of bidder a_j by the same v_{a_1}, and move item t_i from bidder a_i to bidder a_{i+1} for all $i = 1, \dots, j-1$. In this case, the utility of bidders $a_1, a_2, \dots, a_{j-1}$ is unchanged, the utility of bidder a_j increases by $v_{a_j} - v_{a_i} > 0$, and the utility of the auctioneer is unchanged. The sum of payments by the bidders is likewise unchanged. This contradicts the assumption that (H, p) is PO.

We now seek to prove that $(\mathcal{R}_1 \wedge \mathcal{R}_2) \Rightarrow \mathcal{Q}$. We note above that if not all items are allocated ($\neg\mathcal{R}_1$) then the allocation is not PO ($\neg\mathcal{Q}$), thus $\mathcal{Q} \Rightarrow \mathcal{R}_1$ and (trivially) $\mathcal{Q} \Rightarrow \mathcal{Q} \wedge \mathcal{R}_1$ (PO implies that all items allocated). Thus, $(\mathcal{R}_1 \wedge \mathcal{R}_2) \Rightarrow \mathcal{Q} \Rightarrow \mathcal{Q} \wedge \mathcal{R}_1$. If $\mathcal{R}_1$ is false this predicate becomes $\mathrm{False} \Rightarrow$ False, thus we remain with the case where all items are allocated.

We show the contrapositive: $\neg\mathcal{Q} \Rightarrow (\neg\mathcal{R}_1 \vee \neg\mathcal{R}_2)$. Assume $\neg\mathcal{Q}$, i.e., assume that (H, p) is not PO. Further assume $\mathcal{R}_1$, that H assigns all items. We will show $\neg\mathcal{R}_2$, i.e., that there is a trading path with respect to (H, p). Since (H, p) is not PO, there must be some other allocation (H', p') that is not worse for all players (including the auctioneer) and strictly better for at least one player. We can assume that (H', p') assigns all items as well, as otherwise we can take an even better allocation that would assign all items.

By Lemma K.5 we know that H and H' are related by a set of alternating paths (simple and not) and cycles. On a path, the first bidder gives up one item, whereas the last bidder receives one item more, after items are exchanged along the path. Cycles represent giving up one item in return for another by passing items around along it. Cycles do not change the number of items assigned to the bidders along the cycles so we will ignore them. Moreover, by Lemma K.3 we know that every trading path that is not simple has a Pareto equivalent simple trading path and if it does not exist any simple trading path then there are no trading paths at all. Thus, we can focus on the existence of simple trading paths. Let us denote the number of alternating paths by z, and denote the start and end bidders along these z alternating paths by $x_1, \dots, x_z$ and $y_1, \dots, y_z$. We assume that the same bidder may appear multiple times amongst x_i's or multiple times amongst y_i's, but cannot appear both as an x_i and as a y_i, since we can concatenate two such paths into one. Such an alternating path represents a shuffle of items between bidders where bidder x_j loses an item, and bidder y_j gains an item when moving from H to H'. In general, these two items may be entirely different.

Assume there are no trading paths with respect to (H, p). Then it must be the case that for each alternating path j either $v_{y_j} \le v_{x_j}$ holds, $b^*_{y_j} < v_{x_j}$ holds, or both holds, where $b^*_{y_j}$ is the budget left over for bidder y_j at the end of the mechanism. We define $\mu = \{j \in \{1, \dots, z\} | v_{y_j} \le v_{x_j}\}$ and $\nu = \{1, \dots, z\} \setminus \mu$.

Now, no bidder is worse off in (H', p') in comparison to (H, p), the auctioneer is not worse off, and, by assumption, either (A) some bidder is strictly better off, or (B) the auctioneer is strictly better off.

First, we rule out case (B) above: Consider the process of changing (H, p) into (H', p') as a two stage process: at first, the bidders $x_1, \dots, x_z$ give up items. During this first stage, the payments made by these bidders must decrease (in sum) by at least $Z^- = \sum_{i=1}^{z} v_{x_i}$. The second stage is that bidders $y_1, \dots, y_z$ receive their extra items. In the second stage, the maximum extra payment that can be received from bidders $y_1, \dots, y_z$ is no more than

$$Z^+ = \sum_{j \in \mu} v_{y_j} + \sum_{j \in \nu} b^*_{y_j} \le \sum_{j \in \mu} v_{x_j} + \sum_{j \in \nu} v_{x_j} = Z^-, \tag{7}$$

by definition of sets μ and ν above. Thus, the total increase in revenue to the auctioneer would be $Z^+ - Z^- \le 0$. This rules out case B. Moreover, as the auctioneer cannot be worse off, $Z^+ = Z^-$ and from Equation (7) we conclude that

$$\sum_{j\in\mu} v_{y_j} + \sum_{j\in\nu} b^*_{y_j} = \sum_{j\in\mu} v_{x_j} + \sum_{j\in\nu} v_{x_j}. \tag{8}$$

By definition, we have for $j \in \mu$ that $v_{y_j} \le v_{x_j}$, and for $j \in \nu$ we have that $b^*_{y_j} < v_{x_j}$. Thus, if $\nu \neq \emptyset$ then the left hand side of Equation (8) is strictly less than the right hand side, a contradiction. Therefore, case (A) must hold and it must be that $\nu = \emptyset$. We will conclude the proof of the theorem by showing that these two are inconsistent. By (A), we have that $|H'_a|v_a - p'_a = |H_a|v_a - p_a$ for each bidder a whose utility does not increase, $|H'_{\hat{a}}|v_{\hat{a}} - p'_{\hat{a}} > |H_{\hat{a}}|v_{\hat{a}} - p_{\hat{a}}$ for at least one bidder $\hat{a}$, and $\sum_{a\in I} p'_a = \sum_{a\in I} p_a$. We can now derive that

$$\sum_{a\in I} |H'_a|v_a > \sum_{a\in I} |H_a|v_a + \left(\sum_{a\in I} p'_a - \sum_{a\in I} p_a\right),$$

and hence,

$$\sum_{a\in I} (|H'_a| - |H_a|)v_a > 0. \tag{9}$$

Now, whenever $a = x_j$ for a $j \in \{1, \dots, z\}$ we decrease $|H'_a| - |H_a|$ by one, whenever $a = y_j$ for a $j \in \{1, \dots, z\}$ we increase $|H'_a| - |H_a|$ by one. Thus, rewriting Equation (9) we get that

$$\sum_{a\in I} (|\{j \in \{1, \dots, z\} | a = y_j\}| - |\{j \in \{1, \dots, z\} | a = x_j\}|)v_a > 0,$$

respectively

$$\sum_{j=1}^{z} v_{y_j} - \sum_{j=1}^{z} v_{x_j} > 0.$$

Hence,

$$\sum_{j=1}^{z} v_{y_j} > \sum_{j=1}^{z} v_{x_j}, \tag{10}$$

but Equation (10) is inconsistent with Equation (8) as $\nu = \emptyset$ implies that $\mu = \{1, \dots, z\}$. □

We define the auction in Algorithm 4. During the execution of the algorithm there is always a price π (initially zero), a set of unsold items R (i.e., of items with unsold instances) of cardinality $\bar{r} = |R|$, a vector of remaining budgets $b = (b_1, b_2, \dots, b_n)$, and a vector of the number of unsold slots that are instances of the same item $(c_1, c_2, \dots, c_{\bar{r}})$.

Algorithm 4 Combinatorial Auction with Budgets.

procedure COMBINATORIAL AUCTION WITH BUDGETS($v, b, (S_i)_{i \in I}$)
  $\pi \leftarrow 0$
  while $A \neq \emptyset$ **do**
    SELL(E)
    $A \leftarrow A - E$
    repeat
      if $\exists i | B(\neg\{i\}) < \bar{t}$ **then** SELL($\{i\}$)
      else
        For an arbitrary bidder i with $d_i > D_i^+(\pi)$:
          $d_i \leftarrow D_i^+(\pi)$
      end if
    until $\forall i$: $(d_i = D_i^+(\pi)) \wedge (B(\neg\{i\}) \geq \bar{t})$
    Increase π until for some i, $D_i(\pi) \neq D_i^+(\pi)$
  end while
end procedure

Algorithm 5 Compute an Avoid Matching via Min Cost Max Flow.

procedure S-AVOID MATCHING
  Construct interest graph G:
  — Each active bidder $a \in A$ on the left with capacity constraint d_a.
  — Each unsold item $r \in R$ on the right with capacity constraint c_r.
  — Edge (a, r) from bidder $a \in A$ to unsold item $r \in R$ iff $r \in S_a$.
  Return maximal B-matching with minimal number of items assigned to bidders in S, amongst all maximal B-matchings.
end procedure

Algorithm 6 Selling to the Set S of Bidders.

procedure SELL(S)
  repeat
    Compute $Y = S$-AVOID MATCHING
    For arbitrary (a, r) in Y with $a \in S$, sell item r to bidder a and set $S_a \leftarrow S_a \setminus \{r\}$.
  until $B(\neg S) \geq \bar{t}$
end procedure

We denote by U the multiset formed by the multiset-union of the unsold slots of all items and by $\bar{t} = \sum_{i=1}^{\bar{r}} c_i$ its cardinality. The current demand of bidder i during the course of the auction is the number of slots that bidder i could clinch at price π and is denoted by d_i. It is either equal to D_i or to D_i^+, which are defined as follows:

$$D_i(\pi, \bar{r}, b) := \begin{cases} \min\{\bar{r}, |S_i|, \lfloor \frac{b_i}{\pi} \rfloor\}, & \text{if } \pi \leq v_i \\ 0, & \text{else} \end{cases} \qquad D_i^+(\pi, \bar{r}, b) := \lim_{\epsilon \to 0^+} D_i(\pi + \epsilon, \bar{r}, b)$$

We now define the set $A := \{i \in I | D_i > 0\}$ of bidders with positive demand and the subset $E := \{i \in I | D_i > 0 \wedge v_i = \pi\}$ of the bidders in A with valuation equal to the current price. Bidders in A are called *active bidders* whereas bidders in E are called *exiting bidders*.

The auction continues, as long as there is a bidder that belongs to A. At every price π we first try to sell slots to any exiting bidder because even if the utility of the exiting bidder does not increase with the new item, the utility of the auctioneer will. After this check, we have to verify if any bidder can clinch any slot and eventually sell that slot to him. We denote by $B(\{\neg i\})$ the number of slots assigned to bidders other than i in a maximal B-matching, and assign to bidder i the minimal number of items amongst all the maximal B-matchings. An item is clinched by bidder i when $B(\neg\{i\}) < \bar{t}$. If no item is clinched, we set $d_i = D_i^+$ for a bidder i with $d_i > D_i^+$, and this loop continues until no bidder i can clinch an item and $d_i = D_i^+ \; \forall i \in I$. Only now we can raise the price. The interest sets, the vector of the number of unsold slots, and the set of unsold items are updated after every time a bidder clinches. The idea of the auction is to sell slots at the highest possible price such that all slots are sold and there exists no competition between bidders. On the contrary, the existence of a trading path indicates that there exists competition on the assignment of the first slot in the path. Hence, the auction contains no trading path in the final allocation.

We finally present the proof of Pareto optimality for the auction described in Algorithm 4.

Theorem 5.1. *The allocation (H^*, p^*) produced by Algorithm 4 is incentive compatible, individually rational, and Pareto optimal.*

PROOF. We first state the fact that the auction will sell all slots of all keywords. As stated in Theorem K.4, this is a necessary condition for Pareto optimality.

LEMMA K.6. *If the multiset-union of all interest sets $S = \biguplus_{i \in I} S_i$ fulfills $U \subseteq S$, the auction will sell all items.*

At the beginning of the auction the price is zero, and thus, every bidder demands his hole interest set S_i, and all slots can be sold. During the auction the demand of bidder i decreases either when he buys a slot, or when his demand gets updated to D_i^+. The first case does not affect the demand for the unsold slots. In the second case, the usage of the B-matching guarantees that the other bidders demand all the unsold slots at the current price when we decrease i's demand.

□

Now we need to show that there are no trading paths in the final allocation (H^*, p^*) produced by Algorithm 4. Consider the set of all trading paths Σ in the final allocation (H^*, p^*).

Definition K.7. We define for every $\sigma \in \Sigma$:

— Let Y^σ be the S-avoid matching used the first time some item r is sold to some bidder a where (a, r) is an edge along σ. Y^σ is either an E-avoid matching (line 4 of Algorithm 4) or an a-avoid matching for some bidder-item edge (a, r) along σ (line 7 of Algorithm 4).
— If Y^σ is an E-avoid matching, let E^σ be this set of exiting bidders.
— If Y^σ is an a-avoid matching, let a^σ be this bidder.
— Let $F^\sigma \subseteq H^*$ be the set of edges (a, r) such that item r was sold to bidder a at or subsequent to the first time that some item r' was sold to some bidder a' for some edge $(a', r') \in \sigma$. The edge (a', r') is itself in F^σ.
— Let t^σ be the number of unsold instances just before the first time some edge along σ was sold. That is, t^σ is equal to the number of instances matched in F^σ.
— Let π^σ be the price at which Y^σ is computed.
— Let b_a^σ be the remaining budget for bidder a before any item is sold in SELL(E^σ) or SELL(a^σ).

We partition Σ into two classes of trading paths:

— Σ_E is the set of trading paths such that $\sigma \in \Sigma_E$ iff Y^σ is some E^σ-avoid matching used in SELL(E^σ) (line 4 of Algorithm 4).
— $\Sigma_{\neg E}$ is the set of trading paths such that $\sigma \in \Sigma_{\neg E}$ iff Y^σ is some a^σ-avoid matching used in SELL($\{a^\sigma\}$) (line 7 of Algorithm 4).

LEMMA K.8. $\Sigma_E = \emptyset$.

PROOF. We need the following claim:

CLAIM K.9. *Let $\sigma = (a_1, r_2, \ldots, a_{j-1}, r_{j-1}, a_j) \in \Sigma_E$ be a trading path, and let (a_i, r_i) be the last edge belonging to Y^σ along σ. Then the suffix of σ starting at a_i, $(a_i, r_i, \ldots, a_j)$, is itself a trading path.*

PROOF. This follows as the valuation of a_i is equal to the current price π^σ when SELL(E^σ) was executed, and the valuation of a_1 is greater than or equal to π^σ as edge (a_1, r_1) was unsold prior to this SELL(E^σ) and does belong to the final F^σ. □

From the claim above we may assume, without loss of generality, that if $\Sigma_E \neq \emptyset$ then $\exists \sigma \in \Sigma_E$ such that the first edge along σ was also the first edge sold amongst all edges of σ, furthermore, all subsequent edges do not belong to Y^σ.

As no further items will be sold to a bidder $a \in E^\sigma$ after this SELL(E^σ), the number of items assigned to E-type bidders is equal for Y^σ and F^σ. We seek a contradiction to the assumption that Y^σ was an E^σ-avoid matching. Note that the matching F^σ is an E^σ-avoid matching by itself because exactly the number of items assigned to E-type bidders in Y^σ are being sold to them. We now show how to construct from F^σ another matching that assigns less items to E-type bidders.

We show that the number of items assigned to bidder a_1 in F^σ can be reduced by one by giving bidder a_{k+1} item r_k for $k = 1, \ldots, j-1$. This is also a full matching but it remains to show that this does not exceed the capacity constraints d_{a_j} of bidder a_j.

As $d_{a_j} = D_{a_j}$ for all $a \in A$ when SELL(E^σ) is executed, bidder a_j has a remaining budget greater than or equal to v_1 at the conclusion of the auction, and each item assigned to bidder a_j in F^σ is sold to him at a price greater than or equal to $\pi^\sigma = v_1$, it follows that at the time of SELL(E^σ) we have that D_{a_j} is greater than the number of items assigned to a_j in F^σ. Thus, we can increase the number of items allocated to a_j by one without exceeding the demand constraint d_{a_j}.

Now, note that a_j is not an E-type bidder, and the new matching constructed assigns less items to E-type bidders than the matching F^σ. Hence, F^σ is not an E^σ-avoid matching, and in turn neither Y^σ is E^σ-avoid matching. □

We have shown that $\Sigma_E = \emptyset$. It remains to show that $\Sigma_{\neg E} = \emptyset$.

Assume $\Sigma_{\neg E} \neq \emptyset$. Order $\sigma \in \Sigma_{\neg E}$ by the first time at which some edge along σ was sold. We know that this occurs within some SELL($\{a^\sigma\}$) for some a^σ and that $a^\sigma \notin E$. Let us define $\sigma = (a_1, r_1, a_2, r_2, \ldots, a_{j-1}, r_{j-1}, a_j)$ be the last path in this order, and let $e = (a^\sigma, r^\sigma) = (a_i, r_i)$.

Recall that Y^σ is the a^σ-avoid matching used when item r^σ was sold to bidder a^σ. Also, $F^\sigma \subseteq H^*$ is the set of edges added to H^* in the course of the auction from this point on (including the current SELL($\{a_i\}$)).

LEMMA K.10. *Let σ, $a^\sigma = a_i$, and $r^\sigma = r_i$ be as above, then there was another full matching X when Y^σ was computed as an a^σ-avoid matching and X has the following properties:*

(a) The suffix of σ from a_i to a_j:

$$\sigma[a_i, \ldots, a_j] = (a_i, r_i, a_{i+1}, r_{i+1}, \ldots, a_{j-1}, r_{j-1}, a_j),$$

is an alternating path with respect to X (i.e., edges (a_k, r_k) where $i \le k \le j-1$ belong to X).

(b) *The number of items assigned to a_i is equal in X and in Y^σ.*

(c) *The number of items assigned to a_j is equal in X and in F^σ.*

PROOF. We use the notation $M(a)$ for the number of items assigned to bidder a in a matching M. We know that $F^\sigma(a_i) \ge Y^\sigma(a_i)$ since there is otherwise a contradiction because Y^σ is an a_i-avoid matching.

Notice that if $F^\sigma(a_i) = Y^\sigma(a_i)$, it is possible to choose $X = F^\sigma$ and the conditions above follow trivially.

Now, consider the case where $F^\sigma(a_i) > Y^\sigma(a_i)$. Y^σ and F^σ are both matchings that assign all t^σ instances, thus by Lemma K.5 we know that the symmetric difference between the two matchings can by expressed by sets of alternating paths. We consider the smallest such set, i.e., no two alternating paths can be concatenated. By Lemma K.3 we know that we can obtain a Pareto equivalent set of simple alternating paths with respect to F^σ. From the fact that $F^\sigma(a_i) > Y^\sigma(a_i)$, we can obtain $\delta = F^\sigma(a_i) - Y^\sigma(a_i)$ alternating paths that start from a_i. Consider one of this paths, $\tau = (a_i = g_1, s_1, g_2, s_2, \ldots, g_l)$, where g_k are bidders, s_k are items, $(g_k, s_k) \in F^\sigma$, and $(s_k, g_{k+1}) \in Y^\sigma$.

We argue that $\sigma[a_i, \ldots, a_j]$ and τ are bidder disjoint, besides the first bidder a_i. By contradiction, choose u to be the first bidder other to a_i in common between τ and $\sigma[a_i, \ldots, a_j]$. For some $i < k' \le j$ and $1 < k \le l$ we have $u = g_k = a_{k'}$. Let $\sigma^{'}$ be the concatenation of the prefix of σ up to a_i, followed by the prefix of τ up to g_k and followed by the suffix of σ from $g_k = a_{k'}$ to the end.

$$\sigma^{'} = (a_1, r_1, \ldots, a_i = g_1, s_1, g_2, \ldots, g_k = a_{k'}, r_{k'}, a_{k'+1}, \ldots, a_j)$$

This is a trading path in F^σ and no edge is sold before (a_i, r_i) in contradiction with the assumption that σ is the last trading path in the defined order amongst all trading paths. Thus, $\sigma[a_i, \ldots, a_j]$ and τ have the bidder a_i in common and the other bidders along the paths are different. It could be possible that they have some items in common but this is no problem.[2] For any such $\tau = (a_i = g_1, s_1, g_2, s_2, \ldots, g_l)$, we can move item s_k from bidder g_k to bidder g_{k+1} where $1 \le k \le l-1$ without violating the demand of bidder g_l because s_{l-1} was assigned to g_l in Y^σ, and g_l is not the first bidder in another alternating path.

As we can do so for all paths τ, we obtain a new full matching X by applying the swaps of all the alternating paths to F^σ. X assigns to a_i the same number of items as Y^σ and from the fact that a_j does not appear in any τ, the number of items assigned to him is again $F^\sigma(a_j)$. □

COROLLARY K.11. $\Sigma_{\neg E} = \emptyset$.

PROOF. Assume that $\Sigma_{\neg E} \ne \emptyset$, select $\sigma \in \Sigma_{\neg E}$ as in Lemma K.10, and let $a^\sigma = a_i$ and $r^\sigma = r_i$. We now seek to derive a contradiction as follows:

— when Y^σ was computed there was also an alternative full matching Y' with fewer items assigned to bidder a_i, contradicting the assumption that Y^σ is an a_i-avoid matching, or

— we show that the remaining budget of bidder a_j at the end of the auction, $b^*_{a_j}$, has $b^*_{a_j} < v_1$, contradicting the assumption that σ is a trading path.

[2]This is the case because if there are items in common but no bidders unless a_i, the edges that belong to $\sigma[a_i, \ldots, a_j]$ are not modified by any τ, so $\sigma[a_i, \ldots, a_j]$ will be an alternating path with respect to the X we will define and the number of items assigned to bidder a_j does not change for the same reason.

Let X be a matching as in Lemma K.10 and F^σ be as defined in Definition K.7. Also, let $X(a)$, $F^\sigma(a)$, be the number of items assigned to bidder a in the full matchings X, F^σ, respectively.

We consider the following cases regarding d_{a_j} when Y^σ, the a_i-avoid matching, was computed:

(a) $d_{a_j} > X(a_j)$: Like in Lemma K.8, we can decrease the number of items sold to a_i by assigning item r_k to bidder a_{k+1} for $k = i, \ldots, j-1$, without exceeding the demand constraint d_{a_j}.

(b) $d_{a_j} = X(a_j)$: We show that $b^\sigma_{a_j} \le (X(a_j)+1)\pi^\sigma$.

— $D_{a_j} = D^+_{a_j}$: Observe that $X(a_j)$ is smaller than the current number of unsold items $\bar{r}$, and smaller than the cardinality of the interest set S_{a_j} of bidder a_j. This follows because $r_{j-i} \in S_{a_j}$, but no instance of r_{j-1} was sold to bidder a_j and an instance of r_{j-1} is unsold at that time by the definition of σ. Thus,

$$X(a_j) = d_{a_j} = \left\lfloor \frac{b^\sigma_{a_j}}{\pi^\sigma} \right\rfloor > \frac{b^\sigma_{a_j}}{\pi^\sigma} - 1,$$

and hence,

$$b^\sigma_{a_j} < (X(a_j)+1)\pi^\sigma.$$

— $D_{a_j} \ne D^+_{a_j}$: Observe that $a_j \notin E$ as $v_{a_j} > v_{a_i}$ and $a_i \notin E$. As $a_j \notin E$, the only reason that $D_{a_j} \ne D^+_{a_j}$ can be that the remaining budget of bidder a_j, $b^\sigma_{a_j}$, is an integer multiple of the current price π^σ. Then, $D^+_{a_j} = D_{a_j} - 1$ and by the same reason as above $D_{a_j} = \lfloor \frac{b^\sigma_{a_j}}{\pi^\sigma} \rfloor$. As $\lfloor \frac{b^\sigma_{a_j}}{\pi^\sigma} \rfloor = \frac{b^\sigma_{a_j}}{\pi^\sigma}$, it follows that

$$X(a_j) = d_{a_j} \ge D^+_{a_j} = D_{a_j} - 1 = b^\sigma_{a_j}/\pi^\sigma - 1,$$

and hence,

$$b^\sigma_{a_j} \le (X(a_j)+1)\pi^\sigma.$$

Note that the current price $\pi^\sigma < v_{a_i}$ because we assume that a_i was sold r_i as a result of $\textsc{Sell}(\{a_i\})$ where a_i is not an exiting bidder and not of $\textsc{Sell}(E)$. As (a_i, r_i) was the first edge that was sold along σ, either r_1 was sold to a_1 for a price larger than π^σ, or r_1 was sold to a_1 at price π^σ as a result of $\textsc{Sell}(\{a_1\})$ where a_1 is not an exiting bidder. Thus, $\pi^\sigma < v_{a_1}$. By Condition (c) of Lemma K.10 we can deduce that

$$b^\sigma_{a_j} \le (X(a_j)+1)\pi^\sigma = (F^\sigma(a_j)+1)\pi^\sigma.$$

Bidder a_j are sold exactly $F^\sigma(a_j)$ items at a price not lower than π^σ. Hence, at the end of the auction the remaining budget $b^*_{a_j}$ of bidder a_j is lesser than or equal to π^σ. This contradicts the assumption that σ is a trading path since

$$b^*_{a_j} \le \pi^\sigma < v_{a_1}.$$

□

L. PROOF OF THEOREM 6.1

We show the impossibility result for the case of two bidders and two homogeneous items. We use the following lemmata. The first one follows closely the one in [Fiat et al. 2011].

LEMMA L.1. *Consider any auction fulfilling the conditions in Theorem 6.1. If bidder i reports his positive marginal valuations $v_i(1)$ and $v_i(2)$ and wins both items then the payment p_j by bidder j with $j \ne i$ is zero.*

PROOF. Consider the case where bidder j reports $v_j(1) = 0$ and $v_j(2) = 0$. Then any PO auction has to assign both items to bidder i. If any of the items were left unassigned, or would be assigned to bidder j, we could assign it to bidder i without changing any payment. This does not change the utility of bidder j, nor the utility of the auctioneer, but would strictly increase the utility of bidder i.

Bidder rationality implies that bidder i pays at most $v_i(1) + v_i(2)$. Additionally, bidder i has an incentive to report smaller valuations $v'_i(1)$ and $v'_i(2)$ with $p_i > v'_i(1) + v'_i(2) > 0$ if his payment p_i would be positive if he reports truthfully. Thus, IC implies $p_i \leq 0$. For bidder j, it follows from bidder rationality that $p_j \leq 0$. Therefore, it follows from auctioneer rationality that both bidders pay exactly zero.

Consider now the case where both bidders report nonzero valuations. For every instance in which bidder j gets no item it must be that $p_j = 0$. If p_j would be positive, bidder j would report zero valuations. Else, if p_j for that instance would be negative, bidder j would report $v_j(1)$ and $v_j(2)$ in the corresponding instance with zero valuations. Both would contradict IC. □

LEMMA L.2. *Consider any auction fulfilling the conditions in Theorem 6.1. If all marginal valuations are positive,* $v_1(1) > v_2(2)$*, and* $v_2(2) \leq b_1$ *then at least one item will be assigned to bidder 1.*

PROOF. Since all marginal valuations are positive, PO implies that both items have to be assigned to the bidders. Thus, it suffices to show that we cannot assign both items to bidder 2. Assume by contradiction, that both items get assigned to bidder 2. Then $p_1 = 0$ by Lemma L.1. If we increase p_1 by $v_2(2)$, decrease p_2 by $v_2(2)$, and assign one of the items that is assigned to bidder 2 to bidder 1 the following happens: the utility of the auctioneer stays unchanged; the utility of bidder 2 stays unchanged; the utility of bidder 1 increases, since $v_1(1) - v_2(2) > 0$. Thus, it is not PO to assign both items to bidder 2. □

LEMMA L.3. *Consider any auction fulfilling the conditions in Theorem 6.1. If all marginal valuations are positive,* $v_1(2) > v_2(1)$ *and* $v_2(1) + v_2(2) \leq b_1$ *then both items will be assigned to bidder 1.*

PROOF. Since all marginal valuations are positive, PO implies that both items have to be assigned to the bidders. Additionally, Lemma L.2 implies that bidder 1 gets assigned to at least one item. We will now assume by contradiction that one item gets assigned to bidder 2. By bidder rationality, bidder 1 pays at most his reported valuation $v_1(1)$. If his payment p_1 would be greater than $v_2(2)$, bidder 1 would report a valuation $v'_1(1)$ between p_1 and $v_2(2)$. By Lemma L.2 he would get at least one item, and by bidder rationality would pay less. Hence, IC implies that $p_1 \leq v_2(2)$. It follows from the assumptions that $v_2(1) \leq b_1 - p_1$. The valuation of bidder 2 for the item that is assigned to him is smaller than the remaining budget and the marginal valuation $v_1(2)$ of bidder 1. We could increase p_1 by $v_2(1)$, decrease p_2 by $v_2(1)$, and assign both items to bidder 1, which would increase the utility of bidder 1 and make bidder 2 and the auctioneer not worse off. Thus, it is not PO to assign one item to bidder 2. It follows that no item can be assigned to bidder 2. □

Moreover, we will use the following theorem from [Dobzinski et al. 2011]. Please note that their definition of *individual-rationality* corresponds to our definition of *bidder rationality* and their definition of *no-positive-transfers* corresponds to our definition of *auctioneer rationality*.

THEOREM L.4 (THEOREM 4.1 IN [DOBZINSKI ET AL. 2011]). *Let* A *be a deterministic truthful mechanism for* m *items and* 2 *players with known budgets* b_1 *and* b_2

that are generic. Assume that A satisfies PO, individual-rationality, and no-positive-transfers. Then if $v_1 \neq v_2$ the outcome of A coincides with that of the clinching auction.

We are now ready to prove the main theorem.

PROOF THEOREM 6.1. We want to show that there is no incentive compatible, individually rational, and Pareto optimal auction for multiple homogeneous indivisible items and agents with private diminishing marginal valuations and public budgets limits.

Let us assume that we have an auction for bidders with diminishing marginal valuations that is IC, IR, and PO. If we use that auction for bidders with additive valuations that can only report additive valuations, then it is still IR and PO. Moreover, since they have no incentive to report another valuation with diminishing marginal valuations, they have also no incentive to report another additive valuation. Hence, we know from Theorem L.4 that the outcome for additive valuations has to be equal to the outcome of the clinching auction in [Dobzinski et al. 2011] if the budgets are generic.

Consider the case of two bidders and two items such that: $v_1(1) = 5$, $v_1(2) = 5$, $b_1 = 3$, $v_2(1) = 2$, $v_2(2) = 2$, and $b_2 = 11$. Since the marginal valuations of the bidders are constant, the valuations of the bidders are additive. Additionally, the budgets b_1 and b_2 are *generic* following the definition by Dobzinski et al. [2011] where S is a partition of the set of items $J = \{1, 2\}$ to the bidders: $b_1^{2,S} = 3$ for all S, $b_1^{1,S} = 3$ if $|S_1| = 0$, $b_1^{1,S} = -2.5$ if $|S_1| > 0$; $b_2^{2,S} = 11$ for all S, $b_2^{1,S} = 9.5$ if $|S_1| < 2$, $b_2^{1,S} = 11$ if $|S_1| = 2$; thus, for each $k \in \{1, 2\}$ we have that $b_1^{k,S} \neq b_2^{k,S}$ for all S. Thus, both bidders receive one item at prices $p_1 = 2$ and $p_2 = 1.5$ and the utility for the two bidders are $u_1 = 5 - 2 = 3$ and $u_2 = 2 - 1.5 = 0.5$.

Now, assume that the true marginal valuations for bidder 2 are $v_2(1) = 2$ and $v_2(2) = 1$. It follows from Lemma L.3 that all items are assigned to bidder 1. Thus, from Lemma L.1, the payment of bidder 2 will be zero and his utility will be zero too. We conclude that bidder 2 has an incentive to lie about his marginal valuation $v_2(2)$.□