Bidder Optimal Assignments for General Utilities[☆]

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Abstract

We study the problem of matching bidders to items where each bidder i has general, strictly monotonic utility functions $u_{i,j}(p_j)$ expressing his utility of being matched to item j at price p_j. For this setting we prove that a bidder optimal outcome always exists, even when the utility functions are non-linear and non-continuous. We give sufficient conditions under which every mechanism that finds a bidder optimal outcome is incentive compatible. We also give a mechanism that finds a bidder optimal outcome if the conditions for incentive compatibility are satisfied. The running time of this mechanism is exponential in the number of items, but polynomial in the number of bidders.

Keywords: mechanism design, matching markets, discontinuous utilities, envy freeness, lattice

1. Introduction

In matching markets bidders are to be matched to items and the auctioneer receives a monetary compensation from the bidders. Such markets have been studied for several decades (see, e.g., [2]). A practically relevant aspect of such markets is that bidders typically have limited funds. As a concrete example consider the housing market. Here a bidder may be willing to buy a certain house, but he may not be able to pay for it from his own pocket. To buy the house the bidder may decide to take out a loan. The conditions of the loan will depend on its volume and a higher loan will typically incur higher costs. Hence the utility of the bidder will drop faster as the the price of the house increases. This leads to a *non-linear* utility function. In addition, the bank may have a strict limit on the amount of money she is willing to lend the bidder. This strict limit causes an abrupt drop in the bidder's utility at this point and leads to a *discontinuous* utility function. Matching markets like the housing market in which the bidders have both non-linear *and* discontinuous utility functions are the intended application of the model studied here.

In our model n bidders are to be matched to k items. Each bidder i has general utility functions $u_{i,j}(p_j)$ expressing his utility of being matched to item j at price p_j. We allow every bidder i to have an outside option o_i, i.e., a lower bound on the utility u_i that bidder i is guaranteed to get even if he is not matched to any item. We also allow every bidder-item pair (i, j) to have a reserve price $r_{i,j}$, i.e., a lower bound on the price p_j that is required if bidder i is matched to item j. We make three assumptions regarding the utility functions: (i) They are strictly monotonically decreasing in the price. (ii) They drop below the outside options for high enough prices. (iii) They are locally right-continuous. To be specific, the utility functions that we consider here need not be linear and they also need not be continuous.

We are interested in outcomes (μ, p) consisting of a matching μ between bidders and items and prices p. An outcome is *feasible* if the price p_j of every matched item j is at least $r_{i,j}$, where i is the bidder that this item is matched to, and if the utility u_i of every bidder i is at least o_i. An outcome is *envy free* if it is

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feasible and for each bidder i the utility that he gets, i.e., either his outside option or the utility that he gets from being matched to item j at price p_j, is larger than or equal to the utility that he would get if he was matched to any other item k at price p_k. An outcome is *bidder optimal* if it is envy free and if it gives each bidder the highest possible utility among all envy free outcomes. Following earlier work (see, e.g., [3]) we also consider strategic manipulations and analyze *incentive compatible* mechanisms, i.e., mechanisms that ensure that each bidder maximizes his utility by submitting his true utility functions.

Our main result is that a bidder optimal outcome *always exists* if the utility functions satisfy conditions (i), (ii), and (iii) and that such an outcome may not exist if any of these conditions is violated. We establish the existence of a bidder optimal outcome through a *lattice-theoretic argument*: (1) We show that there exists at least one envy free outcome (μ, p). (2) We show that any two envy free outcomes (μ, p) and (μ', p') can be combined into an envy free outcome $(\hat{\mu}, \hat{p})$ with utilities $\hat{u}_i = \max(u_i, u'_i)$ and prices $\hat{p}_j = \min(p_j, p'_j)$. This shows that there are *infimum* envy free prices p^*. (3) We show that there must be a matching μ^* that together with the infimum envy free prices p^* is envy free. This shows that there are *minimum* envy free prices p^*. (4) Finally, we show that an envy free outcome (μ^*, p^*) is bidder optimal if no envy free outcome (μ, p) can have lower prices. This proves the existence of a bidder optimal outcome. The main technical challenge in applying this line of reasoning here is to show that the infimum envy free prices are minimum envy free prices. Our proof of this step is based on a novel argument that consists of lower bounding the price increases that are required to turn a non-envy free outcome into an envy free outcome.

We give *sufficient conditions* under which every mechanism that computes a bidder optimal outcome for utility functions satisfying conditions (i), (ii), and (iii) is *incentive compatible*, and show that bidder optimality need not imply incentive compatibility if any of the sufficient conditions is violated. This in stark contrast to the *continuous* case where bidder optimality always implies incentive compatibility (see, e.g., [4]). We also present two settings with non-linear and discontinuous utility functions in which the conditions for incentive compatibility are satisfied for inputs in *general position* (see, e.g., [3]).[1] In the first setting, the bidders seek to maximize their *return on investment (ROI)*, i.e., valuation divided by price, subject to maximum prices. In the second setting, the bidders are *risk averse*, i.e., the utility for a given item is concave in the price of that item. The utility is concave in the price as higher prices associated with a higher risk of not being able to pay for unforeseen events after the purchase. In both cases an input is said to be in general position if in a certain weighted multi-graph defined on the basis of the utility functions, outside options, and reserve prices no two walks have exactly the same weight. As this is rather unlikely for randomly chosen weights inputs that are not in general position can be regarded as degenerate.

Finally, we give a *mechanism* that computes a bidder optimal outcome for inputs that satisfy the conditions for incentive compatibility. The mechanism is conceptually simple as it takes a brute force approach, but finding a bidder optimal outcome is still not trivial. The reason for this is that while there are only finitely many possible matchings, there are infinitely many prices. Hence we cannot simply check all possible combinations of matchings and prices. The key insight here is that for inputs that satisfy the conditions for incentive compatibility every bidder optimal outcome (μ^*, p^*) satisfies a certain *structural property.* This structural property ensures that the bidder optimal prices p^* can be recovered by (1) removing the matched bidder-item pairs in μ^* in a specific order and by (2) setting the price of the remaining items to the lowest price that ensures envy freeness for the removed bidder. Hence a bidder optimal outcome can be found by checking all possible matchings, all possible orderings of the matched bidder-item pairs in the current matching, and the corresponding prices. We thus obtain a mechanism that runs in time exponential in the number of items, but polynomial in the number of bidders. For applications in which the number of items is constant the running time of this mechanism is polynomial in the input size.

To summarize: (1) We show that a bidder optimal outcome always exists, even if the utility functions are non-linear and non-continuous. (2) We give sufficient conditions under which every mechanism that finds a bidder optimal outcome is incentive compatible. (3) We give a mechanism that finds a bidder optimal

[1]The usage of the expression "general position" is not consistent in the literature. Sometimes general position is defined as algebraic independence with no non-tautological equation relating the input values being satisfied. As we only require certain equations (related to walks in a weighted multi-graph) to be violated this alternate definition of general position is sufficient but not necessary for us.

outcome if the conditions for incentive compatibility are satisfied. The running time of this mechanism is exponential in the number of items, but polynomial in the number of bidders.

2. Related Work

Continuous utility functions. The existence of a bidder optimal outcome for linear utility functions with identical slopes was established in [5], which formulated the problem as a linear program. Later [6] proved that every mechanism that computes a bidder optimal outcome is incentive compatible by showing that the solution to this linear program coincides with the VCG outcome [7, 8, 9]. The classic mechanism for this setting—the Multi-Item Auction of [10]—is based on the Hungarian Method [11]. The basic idea of this mechanism is to start with prices all zero and to iteratively raise the prices of overdemanded items by the same amount. All of these results are for linear and continuous utility functions and are therefore not applicable to the non-linear and discontinuous utility functions studied here.

An incentive compatible, polynomial-time mechanism for piece-wise linear utility functions with non-identical slopes was given in [12, 13]. Just like the Multi-Item Auction this mechanism iteratively raises the prices of overdemanded items, but unlike the Multi-Item Auction it raises the prices by different amounts. Despite the extra generality that piece-wise linear utility functions with non-identical slopes offer, this mechanism is still not applicable to general non-linear and discontinuous utility functions.

The existence of a bidder optimal outcome for general non-linear (but continuous) utility functions was established in [14, 15, 4] and more recently in [16]. While every mechanism that computes a bidder optimal outcome for this setting is incentive compatible (see, e.g, [4]), no mechanism was given that computes an exact bidder optimal outcome. A mechanism that computes an arbitrarily good approximation to the bidder optimal outcome via piece-wise linear approximation of the utility functions was given in [17]. Just like the mechanism presented here this mechanism runs in time polynomial in the number of bidders and exponential in the number of items, but unlike the mechanism presented here this mechanism is only guaranteed to find an approximately bidder optimal outcome for continuous utility functions.

Discontinuous utility functions. Linear utility functions with identical slopes and a single discontinuity were first studied in [3], which gave a polynomial-time, incentive compatible mechanism for inputs in general position. Because an input is not in general position only if two walks in a certain weighted multi-graph have exactly the same weight, which is an unlikely event for randomly chosen weights, such inputs can be considered degenerate. Similar results were subsequently obtained in [18, 19] and [20]. All of these results are only applicable to linear utility functions with identical slopes and a single discontinuity and are therefore not applicable to the non-linear utility functions with multiple discontinuities studied here.

A mechanism that always finds a bidder optimal outcome for piece-wise linear utility functions with non-identical slopes and multiple discontinuities was given in [17]. This mechanism runs in polynomial time and—just like the mechanism of [3]—is incentive compatible for inputs in general position. This mechanism cannot be used to find an exact bidder optimal outcome for general non-linear and discontinuous utility functions and it is not clear whether it can be used to find approximately bidder optimal outcomes for such general, discontinuous utility functions via piece-wise linear approximation.

For a restricted class of discontinuous utility functions, referred to as consistent, an incentive compatible, polynomial-time mechanism was given in [21]. Neither the piece-wise linear utility functions with non-identical slopes and multiple discontinuities studied in [17] nor the general non-linear and discontinuous utility functions studied here are necessarily consistent.

3. Problem Statement

We are given a set I of n bidders and a set J of k items. We use i to denote a bidder and j to denote an item. For each bidder i and item j we are given a *utility function* $u_{i,j}(p_j)$ expressing bidder i's utility for being matched to item j at price p_j. For each bidder i we are given an *outside option* o_i expressing bidder i's utility for being unmatched. For each bidder i and item j we are given a *reserve price* $r_{i,j}$, i.e., a lower bound on the price p_j that bidder i has to pay if he is matched to item j. We make three assumptions concerning the *input* $(u_{i,j}(\cdot), o_i, r_{i,j})$ consisting of the utility functions $u_{i,j}(\cdot)$, the outside options o_i, and the reserve prices $r_{i,j}$: (i) The utility functions $u_{i,j}(\cdot)$ are strictly monotonically decreasing. (ii) For the outside options o_i there exist *threshold values* $\bar{p}_{i,j}$ such that $u_{i,j}(\bar{p}_{i,j}) \leq o_i$. (iii) The utility functions $u_{i,j}(\cdot)$ need not be globally continuous, but they are locally right-continuous, i.e., $\forall x : \lim_{\epsilon \to 0^+} u_{i,j}(x + \epsilon) = u_{i,j}(x)$.

We are interested in *outcomes* (μ, p) consisting of a *matching* $\mu \subseteq I \times J$ and *prices* p. We require that $p_j \geq 0$ for all $j \in J$. We also require that every bidder is matched to at most one item and that every item is matched to at most one bidder. We do *not* require that all bidders and all items are matched. We use $\mu(i)$ to denote the item that bidder i is matched to and $\mu(j)$ to denote the bidder that item j is matched to. Similarly, we use $\mu(I')$ to denote the set of items matched to bidders in $I' \subseteq I$ and $\mu(J')$ to denote the set of bidders matched to items in $J' \subseteq J$. We use u_i to denote bidder i's utility for outcome (μ, p). The utility of bidder i is $u_i = u_{i,\mu(i)}(p_{\mu(i)})$ if he is matched under μ and it is $u_i = o_i$ if he is not matched under μ.

We say that an outcome (μ, p) is *feasible* if (i) $p_j \geq r_{i,j}$ for all bidder-item pairs $(i, j) \in \mu$ and if (ii) $u_i \geq o_i$ for all bidders $i \in I$. The first condition can be interpreted as individual rationality of the auctioneer and the second as individual rationality of the bidders. We say that an outcome (μ, p) is *envy free* if it is feasible and if $u_i \geq u_{i,j}(p_j)$ for all bidder-item pairs $(i, j) \in I \times J$. In other words, an outcome is envy free if no bidder would get a higher utility if he was matched to a different item. We say that an outcome (μ, p) is *bidder optimal* if it is envy free and if $u_i \geq u'_i$ for all bidders $i \in I$ and every envy free outcome (μ', p'), where u_i denotes bidder i's utility for (μ, p) and u'_i denotes his utility for (μ', p'). A bidder optimal outcome thus gives every bidder the highest possible utility among all envy free outcomes.

Even if a mechanism computes a bidder optimal outcome (μ, p) for all inputs $(u_{i,j}(\cdot), r_{i,j}, o_i)$, this does not necessarily mean that a bidder cannot benefit from misreporting his utility functions. A mechanism where this is impossible is said to be *incentive compatible*. More specifically, consider an arbitrary bidder i with outside option o_i and utility functions $u_{i,j}(\cdot)$. Then for every two matrices of utility functions u' and u'' with $u'_{i,j}(\cdot) = u_{i,j}(\cdot)$ for i and all j and $u'_{i',j}(\cdot) = u''_{i',j}(\cdot)$ for all $i' \neq i$ and all j and corresponding outcomes (μ', p') and (μ'', p'') of the mechanism we must have $u'_i \geq u''_i$, where u'_i and u''_i denote bidder i's true utility for (μ', p') and (μ'', p''). Note that this definition does *not* involve the $r_{i,j}$ and the o_i. We assume that the $r_{i,j}$ are a property of the seller and cannot be falsified by the bidders. It is also easy to see that misreporting the o_i is never beneficial to i. Overreporting can only lead to a missed chance of being assigned an item and underreporting can lead to a utility below the true outside option.

4. Existence

We begin by proving the existence of a bidder optimal outcome for discontinuous utility functions. The existence of a bidder optimal outcome for continuous utility functions was established in [14, 15, 4] and more recently in [16]. Our proof is similar to the proof in [4] as it establishes the existence of a bidder optimal outcome through a lattice-theoretic argument. It differs from the proof in [4] in that it does *not* require continuity of the utility functions. At the end of this section we show that all three conditions (i), (ii), and (iii) on the utility functions are required for the existence of a bidder optimal outcome.

Theorem 1. *For all inputs $(u_{i,j}(\cdot), o_i, r_{i,j})$ there exists a bidder optimal outcome (μ^*, p^*).*

The proof strategy is as follows: The first lemma shows that lowest envy free prices are sufficient for bidder optimality. The second lemma shows that any two envy free outcomes (μ, p) and (μ', p') can be combined to give a new envy free outcome $(\hat{\mu}, \hat{p})$ with utilities $\hat{u}_i = \max(u_i, u'_i)$ for all bidders i and prices $\hat{p}_j = \min(p_j, p'_j)$ for all items j. Thus the set of envy free prices $\{p : \exists\ \mu \text{ s.t. } (\mu, p) \text{ is envy free}\}$ has a unique infimum p^*. Lemma 3 shows that the infimum p^* of this set is in fact a minimum, i.e., there exists a matching μ^* that together with the infimum prices p^* is envy free. Lemma 4 finishes the proof as it shows the existence of at least one envy free outcome.

Lemma 1. *If the outcome (μ^*, p^*) is envy free and if $p^*_j \leq p_j$ for all items j and every envy free outcome (μ, p), then the outcome (μ^*, p^*) is bidder optimal.*

Proof. For a contradiction suppose there exists an envy free outcome (μ', p') with $u'_i > u^*_i$ for at least one bidder i. Since (μ^*, p^*) is feasible this implies that $u'_i > u^*_i \geq o_i$, i.e., bidder i must be matched under μ'. Since (μ^*, p^*) is envy free it follows that $u'_i = u_{i,\mu'(i)}(p'_{\mu'(i)}) > u^*_i \geq u_{i,\mu'(i)}(p^*_{\mu'(i)})$ and, thus, $p'_{\mu'(i)} < p^*_{\mu'(i)}$. This contradicts our assumption that $p^*_j \leq p'_j$ for all items j. $\square$

Lemma 2 (Lattice Lemma)**.** *Any two envy free outcomes (μ, p) and (μ', p') can be combined into an envy free outcome $(\hat{\mu}, \hat{p})$ which has $\hat{u}_i = \max(u_i, u'_i)$ for all bidders i and $\hat{p}_j = \min(p_j, p'_j)$ for all items j.*

Proof. Define $I^- = \{i : u_i > u'_i\}$ and $J^+ = \{j : p_j < p'_j\}$.

First we show that $\mu(I^-) \subseteq J^+$ and that $\mu'(I \setminus I^-) \subseteq J \setminus J^+$. Consider any $(i, j) \in \mu$ with $i \in I^-$. Since $u_i = u_{i,j}(p_j) > u'_i \geq u_{i,j}(p'_j)$ it follows that $p_j < p'_j$ and, thus, $j \in J^+$. We conclude that $\mu(I^-) \subseteq J^+$. Consider any $(i, j) \in \mu'$ with $j \in J^+$. Since $u'_i = u_{i,j}(p'_j) < u_{i,j}(p_j) \leq u_i$ it follows that $u'_i < u_i$ and, thus, $i \in I^-$. We conclude that $\mu'(J^+) \subseteq I^-$ or, equivalently, $\mu'(I \setminus I^-) \subseteq J \setminus J^+$.

The matching $\hat{\mu}$ is identical to μ on $I^- \times J^+$ and identical to μ' on $I \setminus I^- \times J \setminus J^+$. This is a valid matching because – as we have just shown – $\mu(I^-) \subseteq J^+$ and $\mu'(I \setminus I^-) \subseteq J \setminus J^+$, i.e., $\mu(I^-) \cap \mu'(I \setminus I^-) = \emptyset$. The prices $\hat{p}_j$ are identical to p_j for all items $j \in J^+$ and identical to p'_j for all items $j \in J \setminus J^+$. Since $p_j < p'_j$ for all items $j \in J^+$ and $p'_j \leq p_j$ for all items $j \in J \setminus J^+$ we have that $\hat{p}_j = \min(p_j, p'_j)$ for all items j.

Next we show that $\hat{u}_i = \max(u_i, u'_i)$ for all bidders i. If $i \in I^-$ then $u_i > u'_i \geq o_i$ shows that i is matched to some item j under μ. Since $\mu(I^-) \subseteq J^+$ we know that $j \in J^+$, i.e., $\hat{p}_j = \min(p_j, p'_j) = p_j$. It follows that $\hat{u}_i = u_{i,j}(\hat{p}_j) = u_{i,j}(p_j) = u_i > u'_i$. If $i \in I \setminus I^-$, then i is either unmatched or matched to some item j under μ'. In the former case, $i \in I \setminus I^-$ implies that $\hat{u}_i = o_i = u'_i \geq u_i$. In the latter case, $\mu'(I \setminus I^-) \subseteq J \setminus J^+$ implies that $j \in J \setminus J^+$ and, thus, $p'_j \leq p_j$. It follows that $\hat{u}_i = u_{i,j}(\hat{p}_j) = u_{i,j}(p'_j) = u'_i \geq u_i$.

The outcome $(\hat{\mu}, \hat{p})$ is feasible because (i) $\hat{p}_j = \min(p_j, p'_j) = p_j \geq r_{i,j}$ for all bidder-item pairs $(i, j) \in \hat{\mu} \cap (I^- \times J^+)$ and $\hat{p}_j = \min(p_j, p'_j) = p'_j \geq r_{i,j}$ for all bidder-item pairs $(i, j) \in \hat{\mu} \cap ((I \setminus I^-) \times (J \setminus J^+))$ and (ii) $\hat{u}_i = \max(u_i, u'_i) \geq o_i$ for all bidders i. It is envy free because (1) $\hat{u}_i = \max(u_i, u'_i) \geq u_i \geq u_{i,j}(p_j) = u_{i,j}(\hat{p}_j)$ for all bidders $i \in I$ and items $j \in J+$ and (2) $\hat{u}_i = \max(u_i, u'_i) \geq u'_i \geq u_{i,j}(p'_j) = u_{i,j}(\hat{p}_j)$ for all bidders $i \in I$ and items $j \in J \setminus J^+$. $\square$

Lemma 3. *If the set of envy free prices $\{p : \exists\ \mu$ s.t. (μ, p) is envy free$\}$ has a unique infimum p^*, then there must be a matching μ^* that together with p^* is envy free.*

The proof of this lemma in [4] (their Property 2) requires continuity of the functions $u_{i,j}(\cdot)$ to deduce that the set of envy free prices is a *closed* set. Closure then implies that the infimum itself is also contained in the set. As we drop the continuity requirement, this line of argument can no longer be used.

Our proof uses the following definitions: Let $I_> = \{i \in I : \max_j u_{i,j}(p_j) > o_i\}$ denote the set of bidders that get a strictly higher utility from being matched to one of their *first choice items* than from their outside option. The *first choice graph* $G_p = (I_> \cup J, F_p)$ at prices p has a node for every bidder $i \in I_>$, a node for every item $j \in J$, and there is an edge between $i \in I_>$ and $j \in J$ if $j \in \mathrm{argmax}_{j'} u_{i,j'}(p_{j'})$. The *feasible first choice graph* $\tilde{G}_p = (I_> \cup J, \tilde{F}_p)$ at prices p has a node for every bidder $i \in I_>$, and a node for every item $j \in J$, and there is an edge between $i \in I_>$ and $j \in J$ if $j \in \mathrm{argmax}_{j'} u_{i,j'}(p_{j'})$ *and* $p_j \geq r_{i,j}$. For bidder $i \in I_>$ and item $j \in J$ we define $F_p(i) = \{j : \exists (i, j) \in F_p\}$ and $F_p(j) = \{i : \exists (i, j) \in F_p\}$. For sets of bidders $T \subseteq I_>$ and sets of items $S \subseteq J$ we define $F_p(T) = \cup_{i \in T} F_p(i)$ and $F_p(S) = \cup_{j \in S} F_p(j)$. We define $\tilde{F}_p(i)$, $\tilde{F}_p(j)$, $\tilde{F}_p(T)$, and $\tilde{F}_p(S)$ analogously. We call a (possibly empty) set of items $S \subseteq J$ *strictly overdemanded* at prices p with respect to a (non-empty) set of bidders $T \subseteq I_>$ if (i) $\tilde{F}_p(T) \subseteq S$ and (ii) $\forall\ R \subseteq S, R \neq \emptyset : |\tilde{F}_p(R) \cap T| > |R|$. We say that the set S is strictly overdemanded at prices p if there exists a set of bidders T such that S is strictly overdemanded at prices p with respect to T.

Using Hall's Theorem [22] we show in Appendix A that (1) there exists an envy free outcome (μ, p) *if and only if* no set of items S is strictly overdemanded at prices p, and (2) if at prices p there exists no matching μ such that the outcome (μ, p) is envy free then there exists a set of bidders T such that the set of items $\tilde{F}_p(T)$ is strictly overdemanded at prices p with respect to T.

Proof of Lemma 3. For a contradiction suppose that for the infimum prices p_* there exists no matching μ^* such that (μ^*, p^*) is envy free. Then, as we show in Appendix A, there must be a set of bidders T such that the set of items $\tilde{F}_{p^*}(T)$ is strictly overdemanded at prices p^* with respect to T.

In *any* envy free outcome $(\hat{\mu}, \hat{p})$ we have $\hat{p}_j \geq p^*_j$ for all items $j \in J$ and, thus, the strict overdemand for the items in $\tilde{F}_{p^*}(T)$ can only be resolved if (i) at least one of the bidders $i \in T$ is happy with his outside option o_i or has a first choice item $j' \in J \setminus F_{p^*}(T)$ under $\hat{p}$ or (ii) for some item $j \in F_{p^*}(T) \setminus \tilde{F}_{p^*}(T)$ we have that $\hat{p}_j \geq r_{i,j}$. Case (i) corresponds, for each pair $(i,j) \in T \times \tilde{F}_{p^*}(T)$, to a price increase relative to p^* of at least $s^i_j = \inf\{x \geq 0 : u_{i,j}(p^*_j + x) \leq o_i \text{ or } u_{i,j}(p^*_j + x) \leq \max_{j' \in J \setminus F_{p^*}(T)} u_{i,j'}(p^*_{j'})\}$, which is strictly larger than zero and contained in the set itself as $u_{i,j}(\cdot)$ is right-continuous.[2] Case (ii) corresponds, for each pair $(i,j) \in T \times F_{p^*}(T) \setminus \tilde{F}_{p^*}(T)$, to a price increase relative to p^* of $f^i_j = r_{i,j} - p^*_j$, which is also strictly larger than zero. Let $\delta^i_j = s^i_j$ if $j \in \tilde{F}_{p^*}(T)$ and let $\delta^i_j = f^i_j$ if $j \in F_{p^*}(i) \setminus \tilde{F}_{p^*}(i)$. Then $\delta = \min_{i \in T, j \in F_{p^*}(T)} \delta^i_j > 0$ is a *lower bound* on the sum of the price increases for *any* envy free outcome $(\hat{\mu}, \hat{p})$.

Lemma 2, however, shows that for any $\epsilon > 0$ there exist envy free prices p' such that $|p'_j - p^*_j| < \epsilon$ for all items j. For $\epsilon = \delta/|J|$ this gives a contradiction to the fact that the price increases corresponding to δ were required by *any* envy free outcome. We conclude that for the infimum prices p^* there exists at least one matching μ^* such that (μ^*, p^*) is envy free. □

Lemma 4. *For all inputs $(u_{i,j}(\cdot), o_i, r_{i,j})$ there exists at least one envy free outcome (μ, p).*

Proof. Let $\mu = \emptyset$ and let $p_j = \max_i(\bar{p}_{i,j})$ for all items $j \in J$, where the $\bar{p}_{i,j}$'s are the threshold values defined above. This outcome is feasible since no item is matched and $u_i = o_i$ for all bidders $i \in I$. It is envy free because $u_i = o_i \geq u_{i,j}(\bar{p}_j) \geq u_{i,j}(p_j)$ for all bidders $i \in I$ and items $j \in J$. □

Proof of Theorem 2. From Lemma 4 we know that there exists at least one envy free outcome (μ, p). By Lemma 2 and Lemma 3 this implies the existence of an envy free outcome (μ^*, p^*) such that $p^*_j \leq p_j$ for all items j and every envy free outcome (μ, p). By Lemma 1 this outcome (μ^*, p^*) is bidder optimal. □

We conclude this section by showing that all three conditions (i)–(iii) on the utility functions (see Section 3) are required to guarantee the existence of a bidder optimal outcome.

Condition (i): There are three bidders and two items. The reserve prices are $r_{i,j} = 0$ for all i and all j and the outside options are $o_i = 0$ for all i. The utility functions are: $u_{1,1}(x) = u_{3,2}(x) = 1 - x$, $u_{1,2}(x) = u_{3,1}(x) = -x$ and $u_{2,1}(x) = u_{2,2}(x) = 2$ if $x \leq 1$ and $u_{2,1}(x) = u_{2,2}(x) = 3 - x$ otherwise. Then one envy free outcome is $\mu = \{(1,1),(2,2)\}$ and $p = (0,1)$ whereas another envy free outcome is $\mu = \{(2,1),(3,2)\}$ and $p = (1,0)$. In neither of the two outcomes can the price for the item with price 1 be lowered any further without upsetting envy freeness. The first outcome is strictly preferred by the first bidder, whereas the second outcome is strictly preferred by the second bidder.

Condition (ii): There are two bidders and one item. Again, the reserve prices are $r_{i,j} = 0$ for all i and all j and the outside options are $o_i = 0$ for all i. The utility functions are $u_{i,1}(x) = 1/(1+x)$ for all i. Then no matter how large p_1 is, both bidders will still strictly prefer the item over being unmatched.

Condition (iii): There are two bidders and one item. As before, the reserve prices are $r_{i,j} = 0$ for all i and all j and the outside options are $o_i = 0$ for all i. The utility functions are $u_{i,1}(x) = 2 - x$ if $x \leq 1$ and $u_{i,1}(x) = -x$ otherwise for all i. Then a price of $p_1 \leq 1$ will not be envy free, as both bidders strictly prefer the item over being unmatched. So any envy free price needs to satisfy $p_1 > 1$ and this set no longer contains its infimum. If we change the first condition of the utility function to $x < 1$, ensuring right-continuity, then the price $p_1 = 1$ is envy free, even though the item cannot be assigned to either of the two bidders.

5. Incentive Compatibility

Next we give sufficient conditions under which every mechanism that computes a bidder optimal outcome is incentive compatible. For this consider input $(u_{i,j}(\cdot), o_i, r_{i,j})$ and denote the bidder optimal utilities for this input by u^*. The conditions are: (a) For every item j there exists a per-item reserve price r_j such that $r_{i,j} = r_j$ for all i. (b) For every *restricted problem* with bidders $I' \subseteq I$, items $J' \subseteq J$, and reserve prices

[2] This no longer holds if the requirement of right-continuity is dropped.

$r'_j = \max(r_j, \max_{i \notin I'} u_{i,j}^{-1}(u_i^*))$ there exists a bidder optimal outcome (μ, p) such that (b1) $p_j = r'_j$ for all unmatched items j and (b2) $p_j = r'_j$ for at least one matched item j if all bidders are matched. At the end of this section we show that each of the conditions is required for incentive compatibility. In Appendix B and Appendix C we present two settings with non-linear and discontinuous utility functions in which the conditions are satisfied for inputs in general position.

Theorem 2. *For all inputs $(u_{i,j}(\cdot), o_i, r_{i,j})$ that satisfy conditions (a) and (b) every mechanism that computes a bidder optimal outcome is incentive compatible.*

We prove this theorem as follows: Consider a set of utility functions $u_{i,j}(\cdot)$. Suppose that for the bidders $i \in I^+$ the $u_{i,j}(\cdot)$ are the true utility functions and that these bidders strictly benefit from reporting utility functions $u'_{i,j}(\cdot)$. Let $u'_{i,j}(\cdot) = u_{i,j}(\cdot)$ for all other bidders $i \in I \setminus I^+$. Then—as we show below—the bidder optimal outcome for $u'_{i,j}(\cdot)$ must be feasible for $u_{i,j}(\cdot)$. If $I^+ = I$, then we get a contradiction from Lemma 5 which shows that no outcome that is feasible for $u_{i,j}(\cdot)$ can give all bidders a strictly higher utility. Otherwise, if $I^+ \subset I$, then we get a contradiction from Lemma 6 which shows that there must be a bidder $i \in I \setminus I^+$ which is *not* envy free with respect to $u_{i,j}(\cdot)$ and, thus, with respect to $u'_{i,j}(\cdot)$.

Lemma 5. *For all inputs $(u_{i,j}(\cdot), o_i, r_{i,j})$ that satisfy conditions (a) and (b) we have that if the outcome (μ^*, p^*) is bidder optimal, then for no feasible outcome (μ', p') we can have $u'_i > u^*_i$ for all i.*

Proof. Since the input satisfies condition (a) there exist per-item reserve prices r_j for all items j such that $r_{i,j} = r_j$ for all bidders i. Since the input satisfies condition (b) there must be a bidder optimal outcome (μ, p) for the original problem such that $p_j = r_j$ for all unmatched items j and $p_j = r_j$ for at least one matched item j if all bidders are matched under μ.

For a contradiction assume that there exists a feasible outcome (μ', p') with $u'_i > u^*_i$ for all bidders i. Since $u_i = u^*_i$ for all bidders i, it follows that $u'_i > u^*_i = u_i$ for all bidders i. Since (μ, p) is feasible this implies that $u'_i > u_i \geq o_i$ for all bidders i and, thus, that (i) all bidders i must be matched under μ'.

Consider an arbitrary bidder-item pair $(i, j) \in \mu'$. Since (μ, p) is envy free, it follows that $u_{i,j}(p'_j) = u'_i > u_i \geq u_{i,j}(p_j)$ and, thus, $p_j > p'_j \geq r_j$. Since $p_j = r_j$ for all items j that are unmatched under μ, this shows that item j must be matched under μ. We conclude that (ii) all items that are matched under μ' are matched under μ and (iii) $p'_j < p_j$ for all of these items j.

From (i) and (ii) we deduce that all bidders are matched under μ. Thus, $p_j = r_j$ for at least one of the items matched under μ. Since the same set of items is matched under μ and under μ' this item j must be matched under μ' and, thus, (iii) shows that $p'_j < p_j = r_j$. We get a contradiction to our assumption that the outcome (μ', p') is feasible. □

Lemma 6. *For all inputs $(u_{i,j}(\cdot), r_{i,j}, o_i)$ that satisfy conditions (a) and (b) we have that if the outcome (μ^*, p^*) is bidder optimal, the outcome (μ', p') is feasible, and $I^+ = \{i \in I \mid u'_i > u^*_i\} \neq \emptyset$, then there exists a bidder-item pair $(i, j) \in I \setminus I^+ \times J$ such that $u'_i < u_{i,j}(p'_j)$.*

Proof. Since the input satisfies condition (a) there exist per-item reserve prices r_j for all items j such that $r_{i,j} = r_j$ for all bidders i. Since the input satisfies condition (b) there must be a bidder optimal outcome (μ, p) for the original problem such that $p_j = r_j$ for all unmatched items j and $p_j = r_j$ for at least one matched item j if all bidders are matched under μ.

Since $u_i = u^*_i$ for all bidders i, we have $I^+ = \{i \in I \mid u'_i > u_i\} \neq \emptyset$. Let $\mu(I^+)$ respectively $\mu'(I^+)$ denote the set of items matched to bidders in I^+ under μ respectively μ'. From Lemma 5 we know that $I^+ \neq I$.

Case 1: $\mu(I^+) \neq \mu'(I^+)$. There must be an item $j \in \mu'(I^+)$ such that $j \notin \mu(I^+)$. Let $i' \in I^+$ be the bidder that is matched to item j under μ'. Since $i' \in I^+$ and the outcome (μ, p) is envy free we have that $u_{i',j}(p'_j) = u'_{i'} > u_{i'} \geq u_{i',j}(p_j)$ and, thus, $p_j > p'_j \geq r_j$. Since $p_j = r_j$ for all items j that are unmatched under μ, this shows that item j must be matched under μ. Let $i \in I \setminus I^+$ be the bidder that is matched to item j under μ. Since $i \notin I^+$ and $p_j > p'_j$ we must have that $u'_i \leq u_i = u_{i,j}(p_j) < u_{i,j}(p'_j)$.

Case 2: $\mu(I^+) = \mu'(I^+)$. Let $J^+ = \mu(I^+) = \mu'(I^+)$. Consider the following *restricted problem*: The set of bidders is I^+, the set of items is J^+, the utility functions are $u^+_{i,j}(\cdot) = u_{i,j}(\cdot)$ for all $(i, j) \in I^+ \times J^+$, the reserve prices are $r^+_j = \max(r_j, \max_{i \notin I^+}(u_{i,j}^{-1}(u_i))$ for all $j \in J^+$, and the outside options are $o^+_i = o_i$ for all

$i \in I^+$. Since the outcome (μ, p) is envy free for the original problem it is also envy free for the restricted problem. It is even bidder optimal because the existence of an envy free outcome (μ'', p'') for the restricted problem in which at least one bidder $i \in I^+$ has a strictly higher utility would imply the existence of an envy free outcome (μ''', p''') for the original problem with this property and therefore contradict the bidder optimality of (μ, p).

Case 2.1: The outcome (μ', p') is feasible for the restricted problem. From Lemma 5 we know that there must be a bidder $i \in I^+$ such that $u'_i \leq u_i$. We get a contradiction to the definition of I^+.

Case 2.2: The outcome (μ', p') is *not* feasible for the restricted problem. Since the outcome (μ', p') is feasible for the original problem this can only happen if for some item $j \in J^+$ we have that $r_j^+ > p'_j \geq r_j$ and, thus, $r_j^+ = \max_{i \notin I^+}(u_{i,j}^{-1}(u_i), 0)$. Since $r_j^+ = 0$ would imply $p'_j < r_j^+ = 0$ this can only happen if $r_j^+ = u_{i,j}^{-1}(u_i)$ for some bidder $i \in I \setminus I^+$. Since $i \in I \setminus I^+$ it follows that $p'_j < r_j^+ = u_{i,j}^{-1}(u_i) \leq u_{i,j}^{-1}(u'_i)$ and, thus, $u'_i < u_{i,j}(p'_j)$. ☐

Proof of Theorem 2. For a contradiction suppose that some subset of bidders $I^+ \subseteq I$ strictly benefits from misreporting their utility functions. Denote the original input by $(u_{i,j}(\cdot), r_j, o_i)$ and the falsified one by $(u'_{i,j}(\cdot), r_j, o_i)$. Note that $u'_{i,j}(\cdot) = u_{i,j}(\cdot)$ for all $(i,j) \in I \setminus I^+ \times J$.

Let (μ^*, p^*) and (μ', p') denote the bidder optimal outcome for the original and falsified input. Denote the utility of bidder i for (μ^*, p^*) and (μ', p') with respect to the original input by u_i^* and u'_i. Denote the utility of bidder i for (μ', p') with respect to the falsified input by u''_i. Note that $I^+ = \{i \in I \mid u'_i > u_i^*\}$.

The outcome (μ', p') is feasible for the original input because (i) $p'_j \geq r_j$ for all items j that are matched under μ' and (ii) $u'_i > u_i^* \geq o_i$ for the bidders $i \in I^+$ and $u'_i = u''_i \geq o_i$ for the bidders $i \in I \setminus I^+$.

Case 1: $I^+ = I$. Lemma 5 shows that no feasible outcome (μ', p') can give all bidders a strictly higher utility than the bidder optimal outcome (μ^*, p^*). This gives a contradiction.

Case 2: $I^+ \neq I$. Lemma 6 shows that if some feasible outcome (μ', p') gives only some of the bidders a strictly higher utility than the bidder optimal outcome (μ^*, p^*), then there must be a bidder $i \in I \setminus I^+$ and an item $j \in J$ for which $u'_i < u_{i,j}(p'_j)$. Since $i \notin I^+$ we have $u''_i = u'_i$ and $u'_{i,j}(\cdot) = u_{i,j}(\cdot)$. It thus follows that $u''_i = u'_i < u_{i,j}(p'_j) = u'_{i,j}(p'_j)$. This contradicts our assumption that the outcome (μ', p') is bidder optimal and therefore envy free for the falsified input. ☐

We conclude this section with three examples that show that for inputs that violate any of the conditions (a), (b1), or (b2) bidder optimality need not imply incentive compatibility.

Condition (a): There are two bidders and two items. The utility functions are $u_{1,1}(x) = 6 - x$, $u_{1,2}(x) = 5 - x$, $u_{2,1}(x) = 6 - x$, and $u_{2,2}(x) = 6 - x$. The reserve prices are $r_{1,1} = 2$, $r_{1,2} = 0$, $r_{2,1} = 1$, and $r_{2,2} = 2$. The outside options are $o_1 = o_2 = 0$. The bidder optimal outcome is $\mu = \{(1,1), (2,2)\}$ and $p = (2,2)$. If the second bidder reports $u_{2,2}(x) = 0 - x$, then the bidder optimal outcome is $\mu = \{(1,2), (2,1)\}$ and $p = (1,0)$. The utility of the second bidder improves from 4 to 5.

Condition (b1): There are two bidders and two items. The utility functions for $i \in \{1,2\}$ are $u_{i,1}(x) = 10 - x$ for $x < 5$, $u_{i,1}(x) = -\infty$ for $x \geq 5$, $u_{i,2}(x) = 1 - x$ for $x < 1$, and $u_{i,2}(x) = -\infty$ for $x \geq 1$. The reserve prices are $r_1 = r_2 = 0$ and the outside options are $o_1 = o_2 = 0$. The bidder optimal outcome is $\mu = \emptyset$ with $p = (5,1)$. If the second bidder reports $u_{2,1}(x) = -\infty$ for $x \geq 0$, then the bidder optimal outcome is $\mu = \{(1,1), (2,2)\}$ and $p = (0,0)$. The utility of the second bidder improves from 0 to 1.

Condition (b2): There are three bidders and three items. The utility functions are: $u_{1,1}(x) = 6 - x$ and $u_{1,2}(x) = 5 - x$ for $x < 6$ and $u_{1,1}(x) = u_{1,2}(x) = -\infty$ otherwise, $u_{2,1}(x) = 11 - x$ and $u_{2,2}(x) = 5 - x$ and $u_{2,3}(x) = 4 - x$ for $x < 4$ and $u_{2,1}(x) = u_{2,2}(x) = u_{2,3}(x) = -\infty$ otherwise, $u_{3,2}(x) = 10 - x$ and $u_{3,3}(x) = 4 - x$ for $x < 3$ and $u_{3,2}(x) = u_{3,3}(x) = -\infty$ otherwise. The reserve prices are $r_1 = r_2 = r_3 = 0$. The outside options are $o_1 = o_2 = 0$. The bidder optimal outcome is $\mu = \{(1,1), (2,2), (3,3)\}$ and $p = (4,3,2)$. If the second bidder reports $u_{2,1}(x) = -\infty$ for $x \geq 0$, then the bidder optimal outcome is $\mu = \{(1,1), (2,3), (3,2)\}$ and $p = (0,1,0)$. The utility of the second bidder improves from 2 to 4.

6. Mechanism

We conclude with a mechanism that computes a bidder optimal outcome for inputs that satisfy conditions (a) and (b) from the previous section. The mechanism is conceptually simple as it takes a brute force approach by trying all possible matchings μ and all possible *orderings* ω of matched bidder-item pairs $(i, j) \in \mu$. For each matching-ordering pair (μ, ω) it keeps lower bounds b_j on the prices p_j that it initializes with r_j. It sets the price p_j of all unmatched items j to $b_j = r_j$. For all unmatched bidders it checks (and aborts) if they envy an unmatched item and it makes sure that they do not envy a matched item by updating the lower bounds b_j of all matched items j. It then considers all matched bidder-item pairs $(i, j) \in \mu$ in the order of ω and sets $p_j = b_j$, checks (and aborts) if bidder i envies an unmatched item or a previously considered matched item, and updates the bounds of all matched items still to come so that bidder i does not experience any envy. For all matching-ordering pairs (μ, ω) for which it does not abort it thus computes a *candidate outcome* (μ, p). After all matching-ordering pairs (μ, ω) have been considered it outputs the candidate outcome that gives each bidder the highest possible utility among all candidate outcomes.

Bidder Optimal Outcome
input: utility functions $u_{i,j}(\cdot)$, reserve prices r_j, outside options o_i
output: bidder optimal outcome (μ^*, p^*)
set $\mu^* = \emptyset$ and $p^*_j = \infty$ for all j
for all matchings μ **do**
  for all possible orderings ω of μ **do**
    set $b_j = r_j$ for all j
    set $p_j = b_j$ for all unmatched j
    for all unmatched bidders i **do**
      /* check whether bidder i envies an unmatched item j */
      if $o_i < u_{i,j}(p_j)$ for some unmatched j, then try next ω
      /* make sure bidder i does not envy a matched item j */
      set $b_j = \max(b_j, u^{-1}_{i,j}(o_i))$ for all matched j
    end for
    for all bidder-item pairs $(i, j) \in \mu$ in the order of ω **do**
      set $p_j = b_j$
      /* check whether it is feasible to match bidder i to item j */
      if $u_{i,j}(p_j) < o_i$, then try next ω
      /* check whether bidder i envies an unmatched item t */
      if $u_{i,j}(p_j) < u_{i,t}(p_t)$ for some unmatched t, then try next ω
      /* check whether bidder i envies a previously considered matched item t */
      if $u_{i,j}(p_j) < u_{i,t}(p_t)$ for some $(s, t) <_\omega (i, j)$, then try next ω
      /* make sure bidder i does not envy a not yet considered matched item t */
      set $b_t = \max(b_t, u^{-1}_{i,t}(u_{i,j}(p_j)))$ for all $(s, t) >_\omega (i, j)$
    end for
  end for
  if $u_i \geq u^*_i$ for all i, then set $\mu^* = \mu$ and $p^*_j = p_j$ for all j
end for
output (μ^*, p^*)

Theorem 3. *For all inputs $(u_{i,j}(\cdot), o_i, r_{i,j})$ that satisfy conditions (a) and (b) the mechanism finds a bidder optimal outcome (μ^*, p^*) in time $O((n + k)^k \cdot k^{2k+1} \cdot n)$.*

We proceed as follows: We first show that for all inputs that satisfy conditions (a) and (b) all bidder optimal outcomes (μ^*, p^*) satisfy a certain structural property and that this structural property induces an ordering ω^* on the matched bidder-item pairs $(i, j) \in \mu^*$. We then use this to show that for all inputs that satisfy conditions (a) and (b) the mechanism computes *at least one* candidate outcome (μ, p) that is bidder optimal. Afterwards we show that for all inputs that satisfy conditions (a) and (b) *all* candidate outcomes (μ, p) computed by the mechanism are envy free.

Lemma 7. *For all inputs $(u_{i,j}(\cdot), o_i, r_{i,j})$ that satisfy conditions (a) and (b), every bidder optimal outcome (μ^*, p^*), and every subset $I^* \times J^* \subseteq \mu^*$ there exists an item $j \in J^*$ for which $p_j^* = \max(r_j, \max_{i \notin I^*} u_{i,j}^{-1}(u_i^*))$.*

Proof. Consider the *restricted problem* with bidders I^*, items J^*, utility functions $u_{i,j}^*(\cdot) = u_{i,j}(\cdot)$, outside options $o_i^* = o_i$, and reserve prices $r_j^* = \max(r_j, \max_{i \notin I^*}(u_{i,j}^{-1}(u_i^*))$. Since the outcome (μ^*, p^*) is envy free for the original problem its restriction to $I^* \times J^*$ is envy free for the restricted problem. It is even bidder optimal because every envy free outcome (μ', p') for the restricted problem induces an envy free outcome (μ'', p'') for the original problem with $u_i'' = u_i^*$ for all $i \notin I^*$ and $u_i'' = u_i'$ for all $i \in I^*$. Hence for every outcome (μ', p') that is envy free for the restricted problem we must have $u_i' \leq u_i^*$ for all bidders $i \in I^*$.

Since the input $(u_{i,j}(\cdot), o_i, r_{i,j})$ satisfies conditions (a) and (b) there exists a bidder optimal outcome (μ', p') for the restricted problem that has $p_j' = r_j^* = \max(r_j, \max_{i \notin I^*} u_{i,j}^{-1}(u_i^*))$ for some item $j \in J^*$. We claim that this item $j \in J^*$ is matched under μ^* and has $p_j^* = p_j'$.

Item j is matched under μ^* because $j \in J^*$ and *all* items in J^* are matched under μ^*. To see that $p_j^* = p_j'$ assume by contradiction that $p_j^* \neq p_j'$. If $p_j^* < p_j'$ then $p_j^* < p_j' = r_j^* = \max(r_j, \max_{i \notin I^*} u_{i,j}^{-1}(u_i^*))$ and we either have $p_j^* < r_j$ or $p_j^* = u_{i,j}^{-1}(u_i^*)$ and, thus, $u_i^* < u_{i,j}(p_j^*)$ for some bidder $i \notin I^*$. We either get a contradiction to the fact that (μ^*, p^*) is feasible or to the fact that (μ^*, p^*) is envy free. If $p_j^* > p_j'$ then for the bidder i that is matched to item j under μ^* we have $u_i^* = u_{i,j}(p_j^*) < u_{i,j}(p_j') \leq u_i'$ because $p_j^* > p_j'$ and (μ', p') is envy free. This contradicts the fact that since (μ^*, p^*) is bidder optimal and (μ', p') is envy free for the restricted problem we must have $u_i^* \geq u_i'$. □

By the previous lemma for all inputs $(u_{i,j}(\cdot), o_i, r_{i,j})$ that satisfy conditions (a) and (b) every bidder optimal outcome (μ^*, p^*) induces an ordering ω^* on the bidder-item pairs $(i, j) \in \mu^*$ as follows:

Induced Ordering
input: bidder optimal outcome (μ^*, p^*)
output: ordering ω^*
$I^* \times J^* = \mu^*$, $\omega^* = \emptyset$
while $I^* \times J^* \neq \emptyset$ **do**
  add $(i, j) \in I^* \times J^*$ for which $p_j^* = \max(r_j, \max_{s \notin I^*} u_{s,j}^{-1}(u_s^*))$ to ω^*
  remove (i, j) from $I^* \times J^*$
end while
output ω^*

Lemma 8. *For all inputs $(u_{i,j}(\cdot), o_i, r_{i,j})$ that satisfy conditions (a) and (b) the mechanism computes at least one candidate outcome (μ, p) that is bidder optimal.*

Proof. Since the input $(u_{i,j}(\cdot), o_i, r_{i,j})$ satisfies conditions (a) and (b) there exists a bidder optimal outcome (μ^*, p^*) such that $p_j^* = r_j$ for all items j unmatched under μ^*. Let $I_{\notin \mu^*}^*$ and $J_{\notin \mu^*}^*$ denote the bidders and items that are unmatched under μ^*. For every ordering ω^* and all m with $0 \leq m \leq |\mu^*|$ let I_m^* and J_m^* denote the first m bidders and items that are matched under μ^*. By Lemma 7 the bidder optimal outcome (μ^*, p^*) induces an ordering ω^* such that for all m with $0 \leq m \leq |\mu^*|$ we have $p_j^* = \max(r_j, \max_{s \in I_{\notin \mu^*}^* \cup I_{m-1}^*} u_{s,j}^{-1}(u_s^*))$, where j is the m-th item matched under μ^*.

We claim that when the mechanism considers μ^* and ω^*, then it computes a candidate outcome (μ, p) with $\mu = \mu^*$ and $p = p^*$. It suffices to show that for all m with $0 \leq m \leq |\mu^*|$ we have (1) $p_j = p_j^*$ for all items $j \in J_{\notin \mu^*}^* \cup J_m^*$ and (2) $u_i = u_i^*$ for all bidders $i \in I_{\notin \mu^*}^* \cup I_m^*$.

For $m = 0$ we have $J_0^* = \emptyset$ and $I_0^* = \emptyset$. Hence the items in $J_{\notin \mu^*}^* \cup J_0^*$ and the bidders in $I_{\notin \mu^*}^* \cup I_0^*$ are precisely the unmatched items $J_{\notin \mu^*}^*$ and bidders $I_{\notin \mu^*}^*$. For every unmatched item $j \in J_{\notin \mu^*}^*$ we have $p_j = b_j = r_j = p_j^*$ (lines 4–5) and for every unmatched bidder $i \in I_{\notin \mu^*}^*$ we have $u_i = o_i = u_i^*$.

For $m > 0$ assume that for all s with $0 \leq s \leq m-1$ we have (1) $p_j = p_j^*$ for all items $j \in J_{\notin \mu^*}^* \cup J_s^*$ and (2) $u_i = u_i^*$ for all bidders $i \in I_{\notin \mu^*}^* \cup I_s^*$. For the m-th matched item j we have $p_j = b_j =$

$\max(r_j, \max_{s \in I^*_{\not\in \mu^*} \cup I^*_{m-1}} u^{-1}_{s,j}(u_s))$ (lines 4, 8, and 15). Since by induction $u_s = u^*_s$ for all $s \in I^*_{\not\in \mu^*} \cup I^*_{m-1}$ this shows that $p_j = \max(r_j, \max_{s \in I^*_{\not\in \mu^*} \cup I^*_{m-1}} u^{-1}_{s,j}(u^*_s)) = p^*_j$. For the m-th matched bidder i we thus have $u_i = u_{i,j}(p_j) = u_{i,j}(p^*_j) = u^*_i$. □

Lemma 9. *For all inputs $(u_{i,j}(\cdot), o_i, r_{i,j})$ that satisfy conditions (a) and (b) all candidate outcomes (μ, p) computed by the mechanism are envy free.*

Proof. Feasibility follows from the fact that $p_j \geq r_j$ for all items j (lines 4–6), that the utility u_i of all unmatched bidders i is o_i by definition, and that for all matched bidders i with $(i, j) \in \mu$ we have $u_i = u_{i,j}(p_j) \geq o_i$ (line 12). Envy freeness follows from the fact that for all unmatched bidders i we have $u_i = o_i \geq u_{i,j}(p_j)$ for all items j (lines 7–8) and that for all matched bidders i with $(i, j) \in \mu$ we have $u_i = u_{i,j}(p_j) \geq u_{i,t}(p_t)$ for all items t (lines 13–15). □

Proof of Theorem 3. For all inputs $(u_{i,j}(\cdot), o_i, r_{i,j})$ that satisfy conditions (a) and (b) the mechanism outputs a bidder optimal outcome (μ^*, p^*) because it computes at least one candidate outcome (μ, p) that is bidder optimal by Lemma 8 and all candidate outcomes (μ, p) are envy free by Lemma 9.

For the running time observe that there are $O((n+k)^k \cdot k^k)$ different matchings of k items to n bidders as there are $\binom{n+k}{k} = O((n+k)^k)$ ways to choose the sets of items and bidders. Observe further that there are up to $k! = O(k^k)$ matchings for a particular choice and up to $k! = O(k^k)$ possible ways of ordering the up to k matched bidder-item pairs. Finally, observe that checking a particular matching-ordering pair takes time $O(nk)$. Hence the total running time is $O((n+k)^k \cdot k^{2k+1} \cdot n)$. □

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Appendix A. Strict Overdemand

Proposition 1. *There exists an envy free outcome (μ, p) if and only if no set of items S is strictly overdemanded at prices p. In particular, if at prices p there exists no matching μ such that the outcome (μ, p) is envy free then there exists a set of bidders T such that the set of items $\tilde{F}_p(T)$ is strictly overdemanded at prices p with respect to T.*

Proof. There exists an envy free outcome (μ, p) if and only if in the feasible first choice graph $\tilde{G}_p = (I_> \cup J, \tilde{F}_p)$ at prices p all bidders can be matched. By Hall's Theorem [22] this is the case if and only if for every set of bidders $T \subseteq I_>$ we have $|\tilde{F}_p(T)| \geq |T|$. We show below that this condition is satisfied if and only if no set of items $S \subseteq J$ is strictly overdemanded with respect to some set of bidders $T \subseteq I_>$.

First suppose that there exists a set of items S that is strictly overdemanded at prices p with respect to some set of bidders T. That is, there exist $S \subseteq J$ and $T \subseteq I_>$ such that (i) $\tilde{F}_p(T) \subseteq S$, and (ii) $\forall\, R \subseteq S, R \neq \emptyset : |\tilde{F}_p(R) \cap T| > |R|$. We claim that $|T| > |\tilde{F}_p(T)|$. If $\tilde{F}_p(T) = \emptyset$ then $|T| > 0$ and $|\tilde{F}_p(T)| = 0$ show that $|T| > |\tilde{F}_p(T)|$. Otherwise, if $\tilde{F}_p(T) \neq \emptyset$, condition (ii) implies that for $R = \tilde{F}_p(T)$ we have that $|\tilde{F}_p(R) \cap T| > |R|$. Since $\tilde{F}_p(R) \cap T = T$ and $R = \tilde{F}_p(T)$ this shows that $|T| > |\tilde{F}_p(T)|$.

Next suppose that there exists a subset of bidders for which the condition from Hall's Theorem is violated. Then there exists $T \subseteq I_>$ such that (a) $|\tilde{F}_p(T)| < |T|$ and (b) $\forall\, T' \subset T : |\tilde{F}_p(T')| \geq |T'|$. We claim that $S = \tilde{F}_p(T)$ is strictly overdemanded at prices p with respect to T. Condition (i) is satisfied because $\tilde{F}_p(T) = S$. To see that condition (ii) is satisfied assume by contradiction that there exists $R \subseteq S, R \neq \emptyset$ such that $|\tilde{F}_p(R) \cap T| \leq |R|$. This implies that for $T' = T \setminus \tilde{F}_p(R) \subset T$ we have that $|\tilde{F}_p(T')| > |T'|$. This gives a contradiction. □

Appendix B. Return on Investment

Consider a setting in which the bidders seek to maximize *return on investment (ROI)*, i.e., valuation divided by price, subject to maximum prices: The utility functions for valuations $v_{i,j} \geq 1$ and maximum prices $m_{i,j} \geq 1$ are $u_{i,j}(p_j) = v_{i,j}/p_j$ if $p_j < m_{i,j}$ and $u_{i,j}(p_j) = -\infty$ otherwise, the outside options are $o_1 \geq 1$, and the reserve prices are $r_j \geq 1$. Any such input $(u_{i,j}(\cdot), o_i, r_j)$ can be transformed into an input $(\bar{u}_{i,j}(\cdot), \bar{o}_i, \bar{r}_j)$ with utility functions $\bar{u}_{i,j}(\bar{p}_j) = \bar{v}_{i,j} - \bar{p}_j = \log(v_{i,j}) - \bar{p}_j$ if $\bar{p}_j < \bar{m}_{i,j} = \log(m_{i,j})$ and $\bar{u}_{i,j}(\bar{p}_j) = -\infty$ otherwise, outside options $\bar{o}_i = \log(o_i)$, and reserve prices $\bar{r}_j = \log(r_j)$.

Proposition 2. *Outcome (μ, p) with $p_j = \exp(\bar{p}_j)$ and $p_j \geq r_j$ is bidder optimal for input $(u_{i,j}(\cdot), o_i, r_j)$ if and only if outcome $(\mu, \bar{p})$ with $\bar{p}_j = \log(p_j)$ and $\bar{p}_j \geq \bar{r}_j$ is bidder optimal for input $(\bar{u}_{i,j}(\cdot), \bar{o}_i, \bar{r}_j)$.*

Proof. For feasibility observe that $p_j \geq r_j$ if and only if $\bar{p}_j \geq \bar{r}_j$ and $u_i \geq o_i$ if and only if $\bar{u}_i \geq \bar{o}_i$. For envy freeness observe that $u_{i,j}(p_j) \geq u_{i,k}(p_k)$ if and only if $\bar{u}_{i,j}(\bar{p}_j) \geq \bar{u}_{i,k}(\bar{p}_k)$. Finally, for bidder optimality observe that $u_i \geq u'_i$ if and only if $\bar{u}_i \geq \bar{u}'_i$. □

For input $(u_{i,j}(\cdot), o_i, r_j)$ define the *input graph* as follows: There is one node per bidder $i \in I$, one node per item $j \in J$, and one dummy item j_0. For every bidder i add a forward edge (i, j_0) with weight $-\log(o_i)$. For every restricted problem with bidders $I' \subseteq I$, items $J' \subseteq J$, and reserve prices r'_j add forward edges (i, j) with weight $-\log(v_{i,j}) + \log(r'_j)$, backward edges (j, i) with weight $\log(v_{i,j}) - \log(r'_j)$, and discontinuity edges (i, j) with weight $\log(m_{i,j}) - \log(v_{i,j}) + \log(r'_j)$. The input is in *general position* if in this input graph no two walks that start with the same bidder, alternate between forward and backward edges, and end with a distinct discontinuity edge have the same weight.

Proposition 3. *If an input $(u_{i,j}(\cdot), o_i, r_j)$ is in general position, then it satisfies conditions (a) and (b).*

Proof. The input satisfies condition (a) because the reserve prices are per item. To see that it satisfies condition (b) consider an arbitrary restricted problem. Since the input is in general position the transformed input associated with the restricted problem is in general position as defined in [17]. Thus, by Lemma 9 in [17], there exist a bidder optimal solution $(\mu, \bar{p})$ for the restricted problem and the transformed input in which (1) $\bar{p}_j = \bar{r}_j$ for all unmatched items j and (2) $\bar{p}_j = \bar{r}_j$ for at least one matched item j. Hence Proposition 2 shows that condition (b) is satisfied for this restricted problem. □

Appendix C. Risk Aversion

Consider a setting with *risk averse* bidders, whose utility gets discounted in a super-linear manner at higher prices. For concreteness suppose that the utility functions for given valuations $v_{i,j} \geq 0$ and maximum prices $m_{i,j} \geq 0$ are $u_{i,j}(p_j) = \log(1 + (v_{i,j} - p_j))$ if $p_j < m_{i,j}$ and $u_{i,j}(p_j) = -\infty$ otherwise, that the outside options are $o_i \geq 0$, and that the reserve prices are $r_j \geq 0$. Any such input $(u_{i,j}, o_i, r_j)$ can be transformed into an input $(\hat{u}_{i,j}(\cdot), \hat{u}_i, \hat{r}_j)$ with utility functions $\hat{u}_{i,j}(p_j) = \hat{v}_{i,j} - p_j = 1 + v_{i,j} - p_j$ if $p_j < \hat{m}_{i,j} = m_{i,j}$ and $\hat{u}_{i,j}(p_j) = -\infty$ otherwise, outside options $\hat{o}_i = \exp(o_i)$, and reserve prices $\hat{r}_j = r_j$.

Proposition 4. *Outcome (μ, p) with $p_j = \hat{p}_j$ and $p_j \geq r_j$ is bidder optimal for input $(u_{i,j}(\cdot), o_i, r_j)$ if and only if outcome $(\mu, \hat{p})$ with $\hat{p}_j = p_j$ and $\hat{p}_j \geq \hat{r}_j$ is bidder optimal for input $(\hat{u}_{i,j}(\cdot), \hat{o}_i, \hat{r}_j)$.*

Proof. For feasibility observe that $p_j \geq r_j$ if and only if $\hat{p}_j \geq \hat{r}_j$ and $u_i \geq o_i$ if and only if $\hat{u}_i \geq \hat{o}_i$. For envy freeness observe that $u_{i,j}(p_j) \geq u_{i,k}(p_k)$ if and only if $\hat{u}_{i,j}(\hat{p}_j) \geq \hat{u}_{i,k}(\hat{p}_k)$. Finally, for bidder optimality observe that $u_i \geq u'_i$ if and only if $\hat{u}_i \geq \hat{u}'_i$. □

For input $(u_{i,j}(\cdot), o_i, r_j)$ define the *input graph* as follows: There is one node per bidder $i \in I$, one node per item $j \in J$, and one dummy item j_0. For every bidder i add a forward edge (i, j_0) with weight $-\exp(o_i)$. For every restricted problem with bidders $I' \subseteq I$, items $J' \subseteq J$, and reserve prices r'_j add forward edges (i, j) with weight $-1 - v_{i,j} + r'_j$, backward edges (j, i) with weight $1 + v_{i,j} - r'_j$, and discontinuity edges (i, j) with weight $m_{i,j} - 1 - v_{i,j} + r'_j$. The input is in *general position* if in this input graph no two walks that start with the same bidder, alternate between forward and backward edges, and end with a distinct discontinuity edge have the same weight.

Proposition 5. *If an input $(u_{i,j}(\cdot), o_i, r_j)$ is in general position, then it satisfies conditions (a) and (b).*

Proof. The argument is the same as in the proof of Proposition 3: Condition (a) is satisfied because the reserve prices are per item. Condition (b) is satisfied for an arbitrary restricted problem because the transformed input associated with the restricted problem is in general position as defined in [17] and, thus, Lemma 9 in [17] and Proposition 4 show that the condition is satisfied. □