Validity in a logic that combines supervaluation and fuzzy logic based theories of vagueness

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Abstract

Supervaluationism and fuzzy logic are two complementary formalisms for reasoning with vague information. We study a framework for combining both approaches. Supervaluationism is modeled by a space of precisifications, essentially a Kripke structure. We equip this space with a probability measure to extract the truth value of each propositional variable by measuring the set of precisifications in which it is true. Complex formulas are evaluated by the truth functions given by a continuous t-norm and its residuum. We also add a universal modality to this logic. Besides unrestricted probability measures, we motivate two other natural classes: strictly positive and uniform probability measures. The goal of this paper is to analyze how the choice of a probability measure and a t-norm affects the set of valid formulas in our hybrid logic.

Keywords: vagueness, supervaluationism, t-norm based logics, mathematical fuzzy logic, non-classical logics

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1. Introduction

Reasoning with vague information is one of the main motivations for fuzzy logic. Another approach for this purpose is supervaluationism and originates in the vagueness discourse in analytic philosophy. Fuzzy logic and supervaluationism follow very different principles. It seems natural to combine these complementary concepts of vagueness to a common framework. In this paper, we study certain aspects of such a framework.

Fuzzy logics are a class of truth-functional logics with the unit interval $[0, 1]$ as the set of truth values. Following Hájek's approach of mathematical fuzzy logic [1], we consider logics that have a continuous t-norm as the truth

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function for conjunction and the corresponding residuum as the truth function for implication. Thus, the choice of a t-norm fully specifies a logic.

The baseline of supervaluationism is that a vague statement should be considered true if it is true for all ways of making it completely precise. Therefore, a vague situation is modeled by a space of precisifications. In every precisification of the space, statements are classically true or false. A person that for example is a borderline case of tallness would be considered tall in some precisifications and not tall in others. The truth in the precisifications should be in accordance with the intuitive use of language: if in a precisification a person with a height of 180 cm is considered tall, then also a person with a height of 190 cm should be considered tall. The supervaluationist's notion of truth is supertruth, which is defined as truth in all precisifications. Note that this model ultimately leads to a Kripke semantics and thus supervaluational logics are usually modal logics.

In this paper, we consider a certain approach of combining supervaluation and fuzzy logics. We extract the truth values of atomic formulas from the Kripke structure of the supervaluational model by equipping it with a probability measure. Complex formulas are interpreted according to the truth functions given by a continuous t-norm. Furthermore, a supertruth operator is added to express truth in all precisifications. Even in this simple framework some natural variations of our combination scheme arise. We could demand that no precisification is measured with 0 or that every precisification is measured uniformly. Thus, there is a certain design choice on how exactly both approaches should be combined. This can be compared to the situation for fuzzy logics where the choice of a t-norm determines properties of the resulting logic. In this paper, we analyze how the choice of the probability measure and the t-norm affects the validity of formulas.

Since the purpose of this paper is to study the effects of *combining* supervaluation and fuzzy logic, we will only work in the simplest possible setting. We restrict ourselves to the propositional level and only consider continuous t-norms and their residua in the truth value interval $[0, 1]$. This means that we do not consider left-continuous t-norms [2] or other generalizations nor any algebraic semantics. Concerning the supervaluational side, we do not impose any accessibility relations on the Kripke structures and assume that the space of precisifications is countable.

1.1. Further motivation

The supervaluational approach towards vagueness is largely motivated by modeling *penumbral connections*. Fine [3] explains that a penumbral connection is a logical relation that holds among indefinite sentences. Fine's example is a (monochrome) blob whose color is at the borderline of red and pink. He argues that the sentence "the blob is red and pink" should be completely false because there can only be one color assigned to the blob. In a truth-functional approach, as for example fuzzy logic, one would usually assign an intermediate truth value, say 0.5, to the sentences "the blob is red" and "the blob is pink." The conjunction of these two sentences would then also receive an intermediate truth-value larger than 0. In the precisification-space approach, all vagueness is resolved in the precisifications. Therefore, in each precisification, exactly one of both sentences is true and the other one is false. Thus, the conjunction "the blob is red and pink" is false in each precisification, i.e., superfalse, which captures Fine's intuition regarding this penumbral connection. Observe also that all classical tautologies are preserved under supertruth.

Supervaluation only needs the qualitative information whether a sentence is true in all precisifications. In the hybrid approach, that was introduced by Fermüller and Kosik [4] and is also pursued in this paper, we additionally want to use the quantitative information conveyed by a precisification space. Intuitively, it should make a difference whether a sentence like "the blob is red" (which is not supertrue) is true in *some* or *almost all* precisifications. In a finite precisification space, this motivates the definition of the truth degree of a sentence as the relative frequency of those precisifications in which the sentence is true. If the sentence "the blob is red" is true in one half of the precisifications and false in the other half, then its truth degree should be 0.5.

As a natural generalization of this idea of extracting truth degrees we can assign weights to the precisifications. This leads to an additive measure whose value for the total space is 1, i.e., a probability measure. Following [5], using a probability measure for this purpose can be motivated as follows. Consider a function μ that assigns to every sentence φ a value $\mu(\varphi)$ which is the degree of belief of a rational agent that φ is true. If $\mu(\varphi)$ really represents this degree of belief, the agent should be willing to accept any bet of the form $(\alpha, \mu(\varphi), \varphi)$ where he has to pay $\alpha\mu(\varphi)$ and receives α if the sentence φ is true and 0 otherwise. In fact, we now describe a situation where the agent will certainly not accept a sequence $(\alpha_i, \mu(\varphi_i), \varphi_i)_{1\leq i\leq m}$ of m such bets. Accepting the bet $(\alpha_i, \mu(\varphi_i), \varphi_i)$ means that the agent has to pay $\alpha\mu(\varphi_i)$ and, in a precisification s, gains α if φ is true in s and 0 otherwise. Thus, the payoff in precisification s is $\alpha_i(\|\varphi_i\|_s - \mu(\varphi_i))$ for the i-th bet, where $\|\varphi\|_s$ is the (classical) truth value of φ in precisification s, and $\sum_i \alpha_i(\|\varphi_i\|_s - \mu(\varphi_i))$ in total. If $\sum_i \alpha_i(\|\varphi_i\|_s - \mu(\varphi_i)) < 0$ for *every* precisification s, then the sequence of bets $(\alpha_i, \mu(\varphi_i), \varphi_i)_{1\leq i\leq m}$ is called a *Dutch book*. A Dutch book implies sure loss for all ways of resolving the vagueness in a precisification space and therefore a rational agent will not accept it. Thus, we are only interested in functions μ that return degrees of belief such that no Dutch book exists. By de Finetti's well-known result [6] we know that there does not exist a Dutch Book against μ if and only if μ is a probability measure (satisfying the Kolmogorov axioms of probability). This fact motivates our use of a probability measure to extract truth values of sentences. To simplify our considerations we define μ as a probability measure directly on the precisifications.

Having extracted truth degrees via a probability measure, we also evaluate the truth degrees of complex combinations of sentences. For this purpose we use the well-known logical connectives derived from a t-norm. Fermüller and Kosik [4] previously considered this approach, but restricted themselves to the Łukasiewicz t-norm. They characterize the valid formulas of the resulting logic as the set of those formulas which a player can assert in a Lorenzen style dialogue and betting game over precisification spaces such that this player does not have to expect any loss of money. It seems natural to generalize their approach from the Łukasiewicz t-norm to arbitrary t-norms.

A second motivation of our hybrid approach is its high expressiveness. Following Kamps analysis [7], we illustrate this circumstance with the comparison "at least as". Consider first the sentence "blob B is at least as red as blob A". One way to formalize this statement is to demand that the implication "if blob A is red, then blob B is red" is supertrue. This works in our framework as it encompasses standard supervaluation. Consider now the sentence "the blob is at least as red as pink". This sentence can be expressed as a fuzzy implication with "the blob is pink" as its antecedent and "the blob is red" as its succedent, which compares the truth degrees of the two statements. Thus, this sentence can easily be formalized in our hybrid approach, but we see no straightforward way of formalizing it in a graded precisification-space approach without truth-functional connectives.

1.2. Related work

Supervaluationism is one of several theories of vagueness that are discussed in analytic philosophy. The formalization of a logic for supervaluationism goes back to Fine's seminal article [3]. Later contributions include Keefe's defense of supervaluationism [8] as well as Shapiro's account of contextualism [9] that has many similarities to supervaluationism. Several newer papers [10, 11, 12, 13] mainly discuss appropriate choices of the entailment relation for a supervaluational logic as well as suitable interpretations of a modality that expresses that a statement is "definitely true".

Our approach of combining supervaluationism and fuzzy logic follows Fermüller and Kosik [4]. They introduced the logic SŁ which is based on the Łukasiewicz t-norm. In this paper, we slightly generalize their model and also consider other t-norms. A similar possibility of combining supervaluationism and fuzzy logic is considered by Bennett [14]. As a possible extension of his standpoint semantics, which are related to the idea of supervaluation, he mentions the possibility of adding a probability measure to a precisification space to extract truth values which then can be handled by t-norm based truth functions.

This idea of equipping a precisification space with a probability measure to extract truth degrees of formulas can be attributed to Kamp [7]. Kamp's precisification-space framework also considers "hypothetical" situations that conflict with the true state of affairs. Kamp discusses linguistic qualifiers like "very" and "rather" as well as comparisons. Edgington [15] also suggests probability measures to extract truth degrees from precisifications, but she argues in favor of logical connectives that are not truth-functional. Lawry and Tang [5] introduce valuation pairs as a model of truth-gaps for propositional sentences. They also consider valuation pairs based on supervaluational principles and extend their approach to bipolar belief measures which are generated from probability distributions on valuation pairs. They justify this approach by a variant of the *Dutch book argument* we mentioned in Section 1.1.

Probabilistic Kripke structures are also used for a "probably"-modality in fuzzy logic [16, 1, 17]. The formula $\mathrm{P}\varphi$ gives the truth degree of the statement that the event described by φ probably happens. The "logic of many" arises from the restriction of using the uniform probability measure on the Kripke structure, which simply counts the relative frequency of events. Technically, our framework enhances this formalism by a (crisp) universal modality, which expresses supertruth. This special case of uniform probabilities is also considered in our contribution.

An approach complementary to ours is Hájek's generalization of Shapiro's machinery [9] to interval-based fuzzy logics [18]. In Hájek's framework, the interpretation of a formula at a precisification is not classical, but based on a t-norm, and every propositional variable receives an interval $[a, b]$ of possible truth values. Precisifying then means to reduce the set of truth values to a subinterval $[c, d] \subseteq [a, b]$. In a completely sharp precisification all truth value intervals of propositional variables collapse to single truth values. In his critique of fuzzy logic as a tool for vagueness, Dubois [19] considers a similar setting in which a vague statement is super-α-true if it is at least α-true in all precisifications.

Another complementary approach is Smith's recent contribution to the vagueness discourse [20], in which he introduces *fuzzy plurivaluationism.* According to Smith, (classical) plurivaluationism expresses "the view that each vague discourse has many acceptable classical interpretations, rather than a unique intended interpretation" [20]. He distinguishes this concept from supervaluationism in which there *is* a unique intended interpretation, namely a partial one and the space of precisifications contains all of its admissible, complete (classical) extensions.[2] In fuzzy plurivaluationism, the acceptable interpretations themselves are fuzzy, each assigning fuzzy truth values to propositions.

The framework of quantitative logic [21] provides a different means of measuring the truth degrees of formulas. The corresponding measure is defined for finite-valued as well as for infinite-valued logical systems. In the case of classical propositional logic, the truth degree of a formula φ measures the proportion of truth value assignments that satisfy φ. The measured truth degrees of formulas are only used "externally" for providing graded versions of basic logical notions such as the consistency of a theory. They are not used "internally" for the evaluation of other formulas, as in our case.

2. Preliminaries

In the following we define all notions needed in this paper and review some basic properties of t-norms.

2.1. Basic definitions

As explained above, the basic idea of our hybrid logic is to measure the "amount of truth" in a precisification space. For this purpose we have to make precise what we mean by measuring. We define the measure in a way that allows

[2]We remark that technically our hybrid model also allows the viewpoint of (classical) plurivaluationism.

us to obtain truth values for propositional variables. The concept that is needed here is that of a probability measure. As a simplification, we restrict ourselves to precisification spaces with only countably many precisifications.[3]

Definition 2.1. A *probability measure* on a countable set S is a function μ from S to the unit interval $[0,1]$ such that $\sum_{s\in S}\mu(s)=1$. To simplify notation we extend μ to subsets of S as follows: $\mu(T)=\sum_{s\in T}\mu(s)$ for every $T\subseteq S$.

As already mentioned, we want to use standard truth functions from fuzzy logic. All these truth function are based on the notion of a continuous t-norm.

Definition 2.2. A continuous t-norm is a continuous function $[0,1]\times[0,1]\to[0,1]$ that is associative, commutative, non-decreasing in both arguments, and has 1 as its neutral element and 0 is its zero element.

The *residuum* $\Rightarrow_*$ of a continuous t-norm $*$ is defined by

$$x \Rightarrow_* y = \max\{z \in [0,1] \mid x * z \leq y\}$$

and its *precomplement* $-_*$ is defined by

$$-_*(x) = (x \Rightarrow_* 0)\ .$$

We intend the t-norm to be the truth function for conjunction, the residuum to be the truth function for implication and the precomplement to be the truth function for negation. Note that for any continuous t-norm $*$, $(x \Rightarrow_* y) = 1$ if and only if $x \leq y$. The three fundamental t-norms are the Łukasiewicz t-norm $x *_{\mathsf{Ł}} y = \max(x+y-1, 0)$, the Gödel t-norm $x *_{\mathsf{G}} y = \min(x,y)$, and the Product t-norm $x *_{\mathsf{P}} y = x \cdot y$.

We now consider precisification spaces that are equipped with a probability measure on the set of precisifications and give appropriate definitions for the truth values of formulas in such a structure.

Definition 2.3. A *precisification space* $\mathbf{S}$ is a triple $\mathbf{S} = \langle \mathbf{P}, (\mathbf{M}_s)_{s\in\mathbf{P}}, \mu\rangle$ that consists of a nonempty, countable set $\mathbf{P}$ of *precisifications*, a function $(\mathbf{M}_s)_{s\in\mathbf{P}}$ that assigns a classical propositional interpretation $\mathbf{M}_s$ to every precisification $s\in\mathbf{P}$, and a probability measure μ on $\mathbf{P}$. As a simplification, we may write $s\in\mathbf{S}$ instead of $s\in\mathbf{P}$. Furthermore, we define the *interpretation of formulas* in a precisification space with an associated continuous t-norm $*$.

The *local* truth value $\|\varphi\|_{s,\mathbf{S}}$ of a formula φ at a precisification $s\in\mathbf{S}$ in a

[3]Technical remark: The restriction to countable precisification spaces simplifies their definition. In light of Proposition 2.9 below this is not a real restriction since infinite precisification spaces can always be reduced to finite ones. The same is true for positive precisification spaces (to be defined below). Uniform precisification spaces, as defined below, are finite anyway.

precisification space $\mathbf{S}$ is inductively defined by:

$$\begin{aligned}
\|\bot\|_{s,\mathbf{S}} &= 0 \\
\|p\|_{s,\mathbf{S}} &= \begin{cases} 1 & \text{if } \|p\|_{\mathbf{M}_s} = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for atomic } p \\
\|\varphi \supset \psi\|_{s,\mathbf{S}} &= \begin{cases} 0 & \text{if } \|\varphi\|_{s,\mathbf{S}} = 1 \text{ and } \|\psi\|_{s,\mathbf{S}} = 0 \\ 1 & \text{otherwise} \end{cases} \\
\|\mathrm{S}\varphi\|_{s,\mathbf{S}} &= \begin{cases} 1 & \text{if } \|\varphi\|_{t,\mathbf{S}} = 1 \text{ for every } t \in \mathbf{S} \\ 0 & \text{otherwise}\,. \end{cases}
\end{aligned}$$

The *global* truth value $\|\varphi\|^*_{\mathbf{S}}$ of a formula φ for a continuous t-norm $*$ and its residuum $\Rightarrow_*$ is inductively defined as follows:

$$\begin{aligned}
\|\bar{0}\|^*_{\mathbf{S}} &= 0 \\
\|p\|^*_{\mathbf{S}} &= \mu\left(\{s \in \mathbf{S} \mid \|p\|_{\mathbf{M}_s} = 1\}\right) \text{ for atomic } p \\
\|\varphi \,\&\, \psi\|^*_{\mathbf{S}} &= \|\varphi\|^*_{\mathbf{S}} * \|\psi\|^*_{\mathbf{S}} \\
\|\varphi \to \psi\|^*_{\mathbf{S}} &= \|\varphi\|^*_{\mathbf{S}} \Rightarrow_* \|\psi\|^*_{\mathbf{S}} \\
\|\mathrm{S}\varphi\|^*_{\mathbf{S}} &= \begin{cases} 1 & \text{if } \|\varphi\|_{s,\mathbf{S}} = 1 \text{ for every } s \in \mathbf{S} \\ 0 & \text{otherwise}\,. \end{cases}
\end{aligned}$$

We consider the formula $\neg\varphi$ as an abbreviation for $\varphi \supset \bot$ if $\neg\varphi$ occurs in the scope of an S-operator and as an abbreviation for $\varphi \to \bar{0}$ otherwise. Note that the definition of the S-operator is similar to the universal modality in Kripke semantics for modal logics.

Using precisification spaces as interpretation structures of formulas, we obtain, for every continuous t-norm $*$, a logic that we call S$*$. For the Łukasiewicz, the Gödel, and the Product t-norm, we call the resulting logics SŁ, SG, and SP, respectively. The notions of truth and validity in such a hybrid logic are defined in the standard way.

Definition 2.4. Let $*$ be a continuous t-norm. A formula φ is *true* for $*$ in a precisification space $\mathbf{S}$ iff $\|\varphi\|^*_{\mathbf{S}} = 1$. A formula φ is *valid* in S$*$ iff φ is true for $*$ in every precisification space $\mathbf{S}$.

The definition above only allows for extracting truth values of atomic formulas. Naturally one would also like to extract truth values of complex formulas. One could for example introduce an additional operator F and define its semantics by $\|\mathrm{F}\varphi\|^*_{\mathbf{S}} = \mu(\{s \in \mathbf{S} \mid \|\varphi\|_{\mathbf{M}_s} = 1\})$. However, this additional operator is not really necessary as the following transformation shows. Consider a formula ψ that contains the subformula $\mathrm{F}\varphi$. Replace all occurrences of $\mathrm{F}\varphi$ in ψ by p (where p is a fresh propositional variable) and call the resulting formula ψ'. Now consider the formula ψ'' defined as $\psi' \,\&\, \mathrm{S}(\varphi \supset p \wedge p \supset \varphi)$[4]. The second part of

[4]The symbol $\wedge$ stands for plain classical conjunction.

ψ'' ensures that p is true at a precisification if and only if φ is true and therefore the measure of p is equal to the measure of φ. Thus ψ is valid if and only if ψ'' is valid.[5]

2.2. Restricted precisification spaces

So far, we considered arbitrary probability measures for precisification spaces. However, one could argue that it makes no sense to give the measure 0 to any precisification because in this case the precisification should not be included in the precisification space anyway. Forbidding precisifications with measure 0 leads to the concept of positive precisification spaces.

Definition 2.5. A precisification space $\mathbf{S}$ with a probability measure μ is *positive* iff $\mu(s) > 0$ for every $s \in \mathbf{S}$. In such a case, μ is called a *positive* probability measure.

In positive precisification spaces the notions of truth and falsehood in terms of truth values and in terms of supertruth and superfalsehood coincide for propositional variables. This is not the case for arbitrary precisification spaces. Thus, another motivation for positive precisification spaces is to prevent that both notions of truth and falsehood come apart for atomic formulas.

Proposition 2.6. *For every positive precisification* $\mathbf{S}$ *and every* p *a propositional variable the following holds:*

- $\|p\|_{\mathbf{S}} = 1$ *if and only if* $\|\mathrm{S}p\|_{\mathbf{S}} = 1$
- $\|p\|_{\mathbf{S}} = 0$ *if and only if* $\|\mathrm{S}\neg p\|_{\mathbf{S}} = 1$

The second restriction that we consider is a natural special case of positive precisification spaces where we give each precisification equal weight. Under this restriction, the local truth value of a propositional variable can be simply determined by counting the number of precisifications at which it is true. Note that a similar restriction has also been considered for the "logic of many" [1].

Definition 2.7. A precisification space $\mathbf{S}$ with a probability measure μ and a finite set of precisifications $\mathbf{P}$ is *uniform* iff $\mu(s) = \frac{1}{|\mathbf{P}|}$ for every $s \in \mathbf{S}$. In such a case, μ is called a *uniform* probability measure. Note that for a uniform space $\mathbf{S}$ we have $\|p\|_{\mathbf{S}} = \frac{|\{s \in \mathbf{P} | \|p\|_{s,\mathbf{S}} = 1\}|}{|\mathbf{P}|}$.

Based on these concepts we now define two restricted forms of validity.

Definition 2.8. Let $*$ be a continuous t-norm and φ a formula. We call φ *p-valid* in $\mathrm{S}*$ iff $\|\varphi\|^*_{\mathbf{S}} = 1$ for every positive precisification space $\mathbf{S}$ and we call φ *u-valid* in $\mathrm{S}*$ iff $\|\varphi\|^*_{\mathbf{S}} = 1$ for every uniform precisification space $\mathbf{S}$. From now on, we refer to the unrestricted notion of validity (see Definition 2.4) as general validity, or g-validity.

[5] The same holds for the notions of p-validity and u-validity introduced in Section 2.2

In the rest of this paper we study the relationship between g-validity, p-validity and u-validity for different choices of the t-norm. Consider the following assertions for an arbitrary formula φ:

(i) φ is g-valid.

(ii) φ is p-valid.

(iii) φ is u-valid.

Note that trivially (i) implies (ii) and (ii) implies (iii). In this paper, we show the following:

- If $*$ is isomorphic to the Łukasiewicz t-norm, then (ii) implies (i).
- If $*$ is not isomorphic to the Łukasiewicz t-norm, then (ii) does not imply (i).
- If $*$ is the Łukasiewicz t-norm, the Gödel t-norm, or the Product t-norm, then (iii) implies (ii).

In particular, this means that the hybrid logic based on the Łukasiewicz t-norm is the only one in which all three notions of validity are equivalent.

It turns out, that our definitions of validity can be simplified a bit. First of all, as pointed out by Fermüller and Kosik [4], the hybrid logic has a certain finite model property.[6]

Proposition 2.9. *A formula φ is g-valid (p-valid) in* $\mathrm{S}*$ *if and only if* $\|\varphi\|^*_{\mathbf{S}} = 1$ *for every (positive) precisification spaces* $\mathbf{S}$ *with a finite set of precisifications.*

The second simplification reduces uniform probability measures to positive, rational measures.

Proposition 2.10. *Let $*$ be a continuous t-norm and φ a formula. Then φ is u-valid in* $\mathrm{S}*$ *if and only if* $\|\varphi\|^*_{\mathrm{S}} = 1$ *for every precisification space with a probability measure μ such that $\mu(s) \in \mathbb{Q}^{>0}$. We call such a precisification space a positive, rational precisification space.*

Proof sketch. For every precisification $s \in \mathbf{S}$ the measure $\mu(s)$ is rational. Let N denote a common denominator of all measures. For the uniform precisification space $\mathbf{S}'$ we create $N \cdot \mu(s)$ copies of every precisification s in which the local truth values are set just like in s. Note that $N \cdot \mu(s)$ is an integer. Clearly, the fraction of the duplicates of s in $\mathbf{S}'$ compared to all precisifications of $\mathbf{S}'$ is $\mu(s)$. This guarantees that the truth values of propositional variables are the same in both spaces. $\square$

[6] The original formulation did not include positive precisification spaces but it is easy to see that the additional claim is also true.

Note that the question whether p-validity and u-validity are equivalent is related to the following more general question: Given a fuzzy logic based on a continuous t-norm and its residuum, are the same formulas valid for the truth value set $[0, 1]$ and the truth value set $[0, 1] \cap \mathbb{Q}$? Apart from Łukasiewicz, Gödel, and Product logic, this seems to be an open problem. An answer has been given for the corresponding algebraic semantics [22]. In our setting, we can simulate every assignment of real truth values to propositional variables by a positive precisification space and we can also simulate every assignment of rational truth values to propositional variables by a positive, rational precisification space, and thus by a uniform precisification space. If we could show that p-validity and u-validity are equivalent in S$*$, it would imply that, for the fuzzy logic based on $*$ and its residuum, the real and the rational semantics are equivalent in terms of valid formulas. Therefore it seems hard to make our results stronger without having any insight on the more general problem.

2.3. Properties of t-norms

In the following we review two properties of t-norms that will be important for our considerations. First of all, it is well-known that every continuous t-norm $*$ is a combination of isomorphic copies of the Łukasiewicz, the Gödel, and the Product t-norm. For a precise formulation of this statement we have to introduce the concepts of an order isomorphism and a generalized ordinal sum [23].

Definition 2.11. Let $[a_1, b_1] \subseteq [0, 1]$ and $[a_2, b_2] \subseteq [0, 1]$ be subintervals of the unit interval. An *order isomorphism* between $[a_1, b_1]$ and $[a_2, b_2]$ is a bijective function $f : [a_1, b_1] \to [a_2, b_2]$ such that $x < y$ if and only if $f(x) < f(y)$.

Theorem 2.12 (Generalized ordinal sum representation)**.** *For every continuous t-norm $*$ there is a countable family $([a_i, b_i], f_i, *_i)_{i \in I}$ with the following properties:*

- *For every $i \in I$, $[a_i, b_i]$ is a subinterval of $[0, 1]$ that is not a singleton.*
- *For all $i, j \in I$ such that $i \neq j$, the intersection $[a_i, b_i] \cap [a_j, b_j]$ is either empty or a singleton.*
- *For every $i \in I$, f_i is an order isomorphism from $[a_i, b_i]$ onto $[0, 1]$.*
- *For every $i \in I$, the t-norm $*_i$ is either equal to the Łukasiewicz t-norm or to the Product t-norm.*
- *The t-norm $*$ can be characterized as follows:*

$$x * y = \begin{cases} f_k^{-1}\left(f_k(x) *_k f_k(y)\right) & \text{if } x, y \in [a_k, b_k] \text{ for some } k \in I \\ \min(x, y) & \text{otherwise} \end{cases} .$$

- *For every $i \in I$ and all x, y with $a_i \leq y < x \leq b_i$ we have*

$$(x \Rightarrow_* y) = f_i^{-1}\left(f_i(x) \Rightarrow_{*_i} f_i(y)\right) .$$

Note that the index set I might be empty, which gives the Gödel t-norm.

The last item is usually not included in the statement of the theorem, but it easily follows from the other parts.

A peculiarity of the Łukasiewicz t-norm is that its residuum is continuous. In fact, this circumstance is characteristic for the Łukasiewicz t-norm (see Corollary 4.5.2 in [23]).

Proposition 2.13. *The residuum $\Rightarrow_*$ of a continuous t-norm $*$ is continuous if and only if $*$ is order isomorphic to the Łukasiewicz t-norm $*_Ł$, i.e., there is an order isomorphism f such that $x * y = f^{-1}(f(x) *_Ł f(y))$ for all $x, y \in [0,1]$.*

3. Validity in restricted precisification spaces

In the following, we will first show that that g-validity, p-validity and u-validity in S$*$ are equivalent when $*$ is isomorphic to the Łukasiewicz t-norm. Our proof heavily relies on a continuous residuum. However, the Łukasiewicz t-norm (up to isomorphism) is the only continuous t-norm with a continuous residuum (see Proposition 2.13). Therefore it is natural to ask whether we can prove any of the equivalences when the residuum is not continuous. It turns out that we can use different arguments to show the equivalence of p-validity and u-validity for two important cases, namely for the Product t-norm and the Gödel t-norm. However, the continuity of the residuum is really necessary for the equivalence of g-validity and p-validity, as we will also show.

3.1. Equivalence of validity and u-validity in SŁ

In SŁ our three variants of validity are equivalent. We prove this by using the fact that the residuum of the Łukasiewicz t-norm is continuous.

Theorem 3.1. *If a formula φ is u-valid in* SŁ*, then φ is also g-valid in* SŁ*.*

Proof. Let φ^* be a formula that is u-valid in SŁ and let $\mathbf{S}$ be a precisification space with a probability measure μ and a finite number of precisifications $\mathbf{P} = \{s_1, \ldots, s_n\}$, which is sufficient due to Proposition 2.9. We have to show that $\|\varphi^*\|_{\mathbf{S}}^{Ł} = 1$.

We define the vector $\vec{\mu} = (\mu_1, \ldots, \mu_n) = (\mu(s_1), \ldots, \mu(s_n))$ which means that $\mu_1, \ldots, \mu_n$ are real numbers that add up to 1. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a sequence of rational numbers $q_i^{(1)}, q_i^{(2)}, \ldots$ such that $\lim_{j\to\infty} q_i^{(j)} = \mu_i$ for every $1 \leq i \leq n$. If $\mu_i = 0$, then we know that $\lim_{k\to\infty} 1/k = 0 = \mu_i$. Thus, we may assume without loss of generality that $q_i^{(j)} > 0$ for $1 \leq i \leq n$ and $j \geq 1$. In vector notation, we have $\lim_{j\to\infty} \vec{q}^{(j)} = \vec{\mu}$ where $\vec{q}^{(j)} = (q_1^{(j)}, \ldots, q_n^{(j)})$ for $j \geq 1$.

The problem with $\vec{q}^{(j)}$ is that its components need not necessarily add up to 1. We fix this by defining a sequence $r_i^{(1)}, r_i^{(2)}, \ldots$ for $1 \leq i \leq n$ by

$$r_i^{(j)} = \frac{q_i^{(j)}}{\sum_{i=1}^{n} q_i^{(j)}}$$

for $j \geq 1$. Then, for $j \geq 1$, we get that $r_i^{(j)}$ is a rational number such that $0 < r_i^{(j)} \leq 1$ and

$$\sum_{i=1}^{n} r_i^{(j)} = \sum_{i=1}^{n} \frac{q_i^{(j)}}{\sum_{i'=1}^{n} q_{i'}^{(j)}} = \frac{1}{\sum_{i'=1}^{n} q_{i'}^{(j)}} \cdot \sum_{i=1}^{n} q_i^{(j)} = 1 \, .$$

We now apply the well-known rules for computing limits of sums and quotients and get

$$\lim_{j\to\infty} r_i^{(j)} = \lim_{j\to\infty} \frac{q_i^{(j)}}{\sum_{i=1}^{n} q_i^{(j)}} = \frac{\lim_{j\to\infty} q_i^{(j)}}{\sum_{i=1}^{n} \lim_{j\to\infty} q_i^{(j)}} = \frac{\mu_i}{\sum_{i=1}^{n} \mu_i} = \frac{\mu_i}{1} = \mu_i \, .$$

In vector notation, we have $\lim_{j\to\infty} \vec{r}^{(j)} = \vec{\mu}$ where $\vec{r}^{(j)} = (r_1^{(j)}, \dots, r_n^{(j)})$ for $j \geq 1$.

For every vector of real numbers $\vec{x} = (x_1, \dots, x_n)$ such that $x_1 + \dots + x_n = 1$ we define the precisification space $\mathbf{S}_{\vec{x}}$ as having the same set of precisifications $\mathbf{P}$ as $\mathbf{S}$ together with the same local truth values and a probability measure $\mu_{\vec{x}}$ that we define by $\mu_{\vec{x}}(s_i) = x_i$ for each $s_i \in \mathbf{P}$. Furthermore, we want to define a certain evaluation function $f_{\varphi^*}(\vec{x})$ that depends on our initial formula φ^*. We inductively define a function $f_\varphi(\vec{x})$ for every formula φ which also gives us the desired function $f_{\varphi^*}(\vec{x})$:

$$\begin{aligned} f_{\bar{0}}\,(\vec{x}) &= 0 \\ f_p\,(\vec{x}) &= \sum_{i=1}^{n} \|p\|_{s_i,\mathbf{S}} \cdot x_i \text{ for atomic } p \\ f_{\mathrm{S}\psi}\,(\vec{x}) &= \|\mathrm{S}\psi\|_{\mathbf{S}} \\ f_{\psi\&\chi}\,(\vec{x}) &= f_\psi\,(\vec{x}) *_{\text{Ł}} f_\chi\,(\vec{x}) \\ f_{\psi\to\chi}\,(\vec{x}) &= f_\psi\,(\vec{x}) \Rightarrow_{\text{Ł}} f_\chi\,(\vec{x}) \; . \end{aligned}$$

Since we have $\|p\|_{s_i,\mathbf{S}_{\vec{x}}} = \|p\|_{s_i,\mathbf{S}}$ for every propositional variable p and $1 \leq i \leq n$ and $\|\mathrm{S}\psi\|_{\mathbf{S}_{\vec{x}}} = \|\mathrm{S}\psi\|_{s_i,\mathbf{S}}$ for every formula $\mathrm{S}\psi$ it is easy to see that

$$f_\varphi\,(\vec{x}) = \|\varphi\|^{\text{Ł}}_{\mathbf{S}_{\vec{x}}} \, .$$

Since we have fixed the formula φ^* and the precisification space $\mathbf{S}$, the expressions $\|p\|_{s_i,\mathbf{S}}$ and $\|\mathrm{S}\psi\|_{s_i,\mathbf{S}}$ are constants in the definition of f_{φ^*}. This means that $f_{\varphi*}$ is a continuous function because $*_{\text{Ł}}$, $\Rightarrow_{\text{Ł}}$, addition and multiplication by a constant are continuous functions.

By our construction of the sequence $\vec{r}^{(1)}, \vec{r}^{(2)}, \dots$ we know that $\mathbf{S}_{\vec{r}^{(j)}}$ is a positive rational precisification space for every $j \geq 0$. Since φ^* is u-valid by assumption we have $\|\varphi\|_{\mathbf{S}_{\vec{r}^{(j)}}} = 1$ for every $j \geq 0$. We now plug everything together and by the fact that f_{φ^*} is continuous we get

$$\begin{aligned} \|\varphi^*\|_{\mathbf{S}} = \|\varphi^*\|_{\mathbf{S}_{\vec{\mu}}} = f_{\varphi^*}\,(\vec{\mu}) &= f_{\varphi^*}\left(\lim_{j\to\infty} \vec{r}^{(j)}\right) \\ &= \lim_{j\to\infty} f_{\varphi^*}\left(\vec{r}^{(j)}\right) = \lim_{j\to\infty} \|\varphi^*\|_{\mathbf{S}_{\vec{r}^{(j)}}} = \lim_{j\to\infty} 1 = 1 \, . \end{aligned}$$

Since $\mathbf{S}$ was an arbitrary finite precisification space we conclude that φ^* is g-valid. $\square$

Note that the equivalence of g-validity and u-validity in SŁ has the further advantage that validity has been reduced to a finitary notion in which real-valued probability measures do not have to be considered.

3.2. Characterization of the equivalence of validity and p-validity

In the following, we give for every continuous t-norm $*$ that is not isomorphic to the Łukasiewicz t-norm a counterexample formula that is p-valid in S$*$ but not g-valid in S$*$. Remember that a continuous t-norm is isomorphic to the Łukasiewicz t-norm if and only if its residuum is continuous (see Proposition 2.13). An important class of continuous t-norms with non-continuous residua are those continuous t-norms whose precomplement is *Gödel negation* which is the function given by

$$-_{\mathsf{G}}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} .$$

For example, the Gödel t-norm and the Product t-norm both have Gödel negation as their precomplement. Our strategy is to distinguish between those continuous t-norms that have Gödel negation as their precomplement and those that have not. For the first case it is relatively easy to find a counterexample. The second case needs a more involved analysis. There we exploit the fact that all such t-norms "start" with an isomorphic copy of the Łukasiewicz t-norm in the generalized ordinal sum representation (see Theorem 2.12).

Lemma 3.2. *If the precomplement $-_*$ of a continuous t-norm $*$ is Gödel negation $-_{\mathsf{G}}$, then g-validity and p-validity are not equivalent in* S$*$*.*

Proof. The main idea is that Gödel negation allows us to check whether the truth value of a formula is strictly greater than 0 and that for positive precisification spaces, we can enforce that a propositional variable p receives a truth value strictly greater than 0. Let $*$ be a continuous t-norm with Gödel negation and define the formula φ as

$$(\neg \mathrm{S}\neg p) \to (\neg\neg p)\,.$$

We refer to $\neg \mathrm{S}\neg p$ as the antecedent of φ and to $\neg\neg p$ as the succedent of φ. Since $*$ has Gödel negation we have $\|\neg\neg p\|^*_{\mathbf{S}} = \lceil \|p\|_{\mathbf{S}} \rceil$ (0 if $\|p\|_{\mathbf{S}} = 0$, 1 otherwise) for every precisification space $\mathbf{S}$.

We first show that φ is p-valid. Let $\mathbf{S}$ be an arbitrary positive precisification space with a probability measure μ. For the antecedent of φ we know that $\|\neg \mathrm{S}\neg p\|_{\mathbf{S}} \in \{0,1\}$. If the truth value is 0, then trivially $\|\varphi\|^*_{\mathbf{S}} = 1$. Assume now that $\|\neg \mathrm{S}\neg p\|_{\mathbf{S}} = 1$. Since $\mathbf{S}$ is a positive precisification space we may apply Proposition 2.6 and get $\|p\|_{\mathbf{S}} > 0$. Therefore $\|\neg\neg p\|^*_{\mathbf{S}} = \lceil \|p\|_{\mathbf{S}} \rceil = 1$ which means that $\|\varphi\|^*_{\mathbf{S}} = 1$. Because $\mathbf{S}$ was an arbitrary positive precisification space, φ is p-valid.

Finally, we show that φ is not g-valid. Consider the precisification space $\mathbf{S}$ consisting of two precisifications s_1 and s_2 with a probability measure μ given by $\mu(s_1) = 1$ and $\mu(s_2) = 0$. We define the interpretation of the propositional variable p in the precisifications as follows: $\|p\|_{s_1,\mathbf{S}} = 0$ and $\|p\|_{s_2,\mathbf{S}} = 1$. Then $\|p\|_{\mathbf{S}} = 0$ and thus we have $\|\neg\neg p\|^*_{\mathbf{S}} = 0$ for the succedent of φ. For the antecedent of φ we have $\|\neg \mathrm{S}\neg p\|_{\mathbf{S}} = 1$ because $\|\mathrm{S}\neg p\|_{\mathbf{S}} = 0$ due to $\|p\|_{s_2,\mathbf{S}} = 1$. Therefore $\|\varphi\|^*_{\mathbf{S}} = 0$ and thus φ is not g-valid in S$*$ for *any* continuous t-norm $*$. □

Lemma 3.3. *Let $*$ be a continuous t-norm such that the residuum $\Rightarrow_*$ is* not *continuous. If the precomplement $-_*$ is* not *Gödel negation, then g-validity and p-validity are not equivalent in* S$*$.

Proof. The following fact about continuous t-norms is well-known (compare Lemma A.1 and Proposition A.1 in [2]): if the precomplement $-_*$ is *not* Gödel negation, then the t-norm $*$ is isomorphic to the Łukasiewicz t-norm on the first interval $[0, u]$ in the generalized ordinal sum representation (with $u > 0$). Furthermore it must be the case that $u < 1$ because otherwise $*$ would be isomorphic to the Łukasiewicz t-norm on the complete unit interval and thus have a continuous residuum $\Rightarrow_*$ which contradicts our assumption.

We can now define a formula φ that is p-valid but not g-valid. As in the previous proof, the main idea is to use Proposition 2.6 to enforce that p has a truth value greater than 0. Define φ as the following formula:

$$(\neg \mathrm{S}\neg p) \to (\neg\neg q \to (\neg p \to q)) .$$

We refer to $\neg \mathrm{S}\neg p$ as the antecedent of φ and to $\neg\neg q \to (\neg p \to q)$ as the succedent of φ.

We first show that φ is p-valid. Let $\mathbf{S}$ be an arbitrary positive precisification space. Assume that for the antecedent of φ we have $\|\neg \mathrm{S}\neg p\|_{\mathbf{S}} = 1$. Since $\mathbf{S}$ is a positive precisification space this implies $\|p\|_{\mathbf{S}} > 0$. We have to show that the succedent of φ also has the truth value 1.

Consider first the case that $\|p\|_{\mathbf{S}} > u$. We now want to argue that $\|\neg p\|^*_{\mathbf{S}} = 0$. By the definition of the residuum of $*$ we have

$$\|\neg p\|^*_{\mathbf{S}} = \|p \to \bar{0}\|^*_{\mathbf{S}} = (\|p\|_{\mathbf{S}} \Rightarrow_* 0) = \max \{z \in [0, 1] \mid \|p\|_{\mathbf{S}} * z \leq 0\} .$$

By the generalized ordinal sum representation, $\|p\|_{\mathbf{S}}$ lies in an interval $[a, b]$ such that the continuous t-norm $*$ restricted to $[a, b]$ is isomorphic to either the Łukasiewicz or the Product t-norm. Because the intervals of this representation do not overlap and $\|p\|_{\mathbf{S}} > u$ we know that $a \geq u$. If $z \in [a, b]$, then also $\|p\|_{\mathbf{S}} * z \in [a, b]$ and therefore $\|p\|_{\mathbf{S}} * z \geq u > 0$. If $z \notin [a, b]$ and $z > 0$, then $\|p\|_{\mathbf{S}} * z = \min(\|p\|_{\mathbf{S}}, z) > 0$ because $\|p\|_{\mathbf{S}} > u > 0$. Therefore the residuum can only have the value $z = 0$ (for which we get $\|p\|_{\mathbf{S}} * z = 0$). Thus, we have $\|\neg p\|^*_{\mathbf{S}} = 0$ which implies $\|\neg p \to q\|^*_{\mathbf{S}} = 1$ and $\|\neg\neg q \to (\neg p \to q)\|^*_{\mathbf{S}} = 1$.

Consider now the case that $\|p\|_{\mathbf{S}} \leq u$. We have to distinguish two subcases: either $\|q\|_{\mathbf{S}} \geq u$ or $\|q\|_{\mathbf{S}} < u$. Assume that $\|q\|_{\mathbf{S}} \geq u$. Since $0 < \|p\|_{\mathbf{S}} \leq u$ we know by Theorem 2.12 that $\|\neg p\|^*_{\mathbf{S}} = \|p \to \bar{0}\|^*_{\mathbf{S}} \in [0, u]$. This gives $\|\neg p\|^*_{\mathbf{S}} \leq u \leq \|q\|_{\mathbf{S}}$. Therefore $\|\neg p \to q\|^*_{\mathbf{S}} = 1$ and thus $\|\neg\neg q \to (\neg p \to q)\|^*_{\mathbf{S}} = 1$.

Assume that $\|q\|_{\mathbf{S}} < u$. If $\|q\|_{\mathbf{S}} = 0$, then $\|\neg\neg q\|^*_{\mathbf{S}} = 0$ and therefore $\|\neg\neg q \to (\neg p \to q)\|^*_{\mathbf{S}} = 1$. Thus we assume in the following that $\|q\|_{\mathbf{S}} > 0$. If $\|\neg p\|^*_{\mathbf{S}} \leq \|q\|_{\mathbf{S}}$, then $\|\neg p \to q\|^*_{\mathbf{S}} = 1$ and therefore $\|\neg\neg q \to (\neg p \to q)\|^*_{\mathbf{S}} = 1$. Hence we assume in the following that $\|\neg p\|^*_{\mathbf{S}} > \|q\|_{\mathbf{S}}$.

We are now left with the following situation: $0 < \|p\|_{\mathbf{S}} \leq u$, $0 < \|q\|_{\mathbf{S}} < u$, and $\|q\|_{\mathbf{S}} < \|\neg p\|^*_{\mathbf{S}}$. We will apply Theorem 2.12 several times to calculate the truth value of $\neg\neg q \to (\neg p \to q)$. As argued above, the t-norm $*$ is isomorphic to the Łukasiewicz t-norm on the interval $[0, u]$. Let f denote the order isomorphism between $[0, u]$ and $[0, 1]$ as given by the generalized ordinal sum representation (see Theorem 2.12). Note that the residuum of the Łukasiewicz t-norm is given by $x \Rightarrow_{\text{Ł}} y = \min(1 - x + y, 1)$. First of all, since $\|p\|_{\mathbf{S}} > 0$, we have

$$\begin{aligned}
\|\neg p\|^*_{\mathbf{S}} = \|p \to \bar{0}\|^*_{\mathbf{S}} &= f^{-1}\left(\min\left(1 - f\left(\|p\|_{\mathbf{S}}\right) + f(0), 1\right)\right) \\
&= f^{-1}\left(\min\left(1 - f\left(\|p\|_{\mathbf{S}}\right) + 0, 1\right)\right) \\
&= f^{-1}\left(\min\left(1 - f\left(\|p\|_{\mathbf{S}}\right), 1\right)\right) \\
&= f^{-1}\left(1 - f\left(\|p\|_{\mathbf{S}}\right)\right) \in [0, u]
\end{aligned}$$

and since $\|q\|_{\mathbf{S}} > 0$ we have

$$\|\neg q\|^*_{\mathbf{S}} = f^{-1}\left(1 - f\left(\|q\|_{\mathbf{S}}\right)\right) .$$

Now because $\|\neg p\|^*_{\mathbf{S}} > \|q\|_{\mathbf{S}}$ we get

$$\begin{aligned}
\|\neg p \to q\|^*_{\mathbf{S}} &= f^{-1}\left(\min\left(1 - f\left(\|\neg p\|^*_{\mathbf{S}}\right) + f\left(\|q\|_{\mathbf{S}}\right), 1\right)\right) \\
&= f^{-1}\left(\min\left(1 - f\left(f^{-1}\left(1 - f\left(\|p\|_{\mathbf{S}}\right)\right)\right) + f\left(\|q\|_{\mathbf{S}}\right), 1\right)\right) \\
&= f^{-1}\left(\min\left(1 - \left(1 - f\left(\|p\|_{\mathbf{S}}\right)\right) + f\left(\|q\|_{\mathbf{S}}\right), 1\right)\right) \\
&= f^{-1}\left(\min\left(f\left(\|p\|_{\mathbf{S}}\right) + f\left(\|q\|_{\mathbf{S}}\right), 1\right)\right) \in [0, u] .
\end{aligned}$$

Since $\|q\|_{\mathbf{S}} > 0$ and f is an order isomorphism, we get $f(\|q\|_{\mathbf{S}}) > f(0)$. Therefore $1 - f(\|q\|_{\mathbf{S}}) > 0$ and thus $\|\neg q\|^*_{\mathbf{S}} = f^{-1}(1 - f(\|q\|_{\mathbf{S}})) > f^{-1}(0) = 0$. This means that we may apply Theorem 2.12 again and we get

$$\|\neg\neg q\|^*_{\mathbf{S}} = f^{-1}\left(1 - f\left(\|\neg q\|^*_{\mathbf{S}}\right)\right) = \|q\|_{\mathbf{S}} .$$

Since $\|p\|_{\mathbf{S}} > 0$ and $\|q\|_{\mathbf{S}} < u$ we have $f(\|p\|_{\mathbf{S}}) > f(0) = 0$ and $f(\|q\|_{\mathbf{S}}) < f(u) = 1$. Therefore the inequality

$$f\left(\|q\|_{\mathbf{S}}\right) < \min\left(f\left(\|p\|_{\mathbf{S}}\right) + f\left(\|q\|_{\mathbf{S}}\right), 1\right)$$

holds. Since f is an order isomorphism we conclude

$$\|\neg\neg q\|^*_{\mathbf{S}} = f^{-1}\left(f\left(\|q\|_{\mathbf{S}}\right)\right) < f^{-1}\left(\min\left(f\left(\|p\|_{\mathbf{S}}\right) + f\left(\|q\|_{\mathbf{S}}\right), 1\right)\right) = \|\neg p \to q\|^*_{\mathbf{S}} .$$

Therefore we get $\|\neg\neg q \to (\neg p \to q)\|^*_{\mathbf{S}} = 1$.

We have showed for the succedent of φ that $\|\neg\neg q \to (\neg p \to q)\|^*_{\mathbf{S}} = 1$ in all possible cases which means that $\|\varphi\|^*_{\mathbf{S}} = 1$. Since $\mathbf{S}$ was an arbitrary positive probability space, we conclude that φ is p-valid.

Finally, we show that φ is not g-valid. Consider the precisification space $\mathbf{S}$ consisting of three precisifications s_1, s_2 and s_3 with the probability measure μ given by $\mu(s_1) = 0$, $\mu(s_2) = u$, and $\mu(s_3) = 1 - u$. The propositional variables are interpreted at the precisifications as follows:

$$\begin{array}{lll} \|p\|_{s_1,\mathbf{S}} = 1 & \qquad \|p\|_{s_2,\mathbf{S}} = 0 & \qquad \|p\|_{s_3,\mathbf{S}} = 0 \\ \|q\|_{s_1,\mathbf{S}} = 0 & \qquad \|q\|_{s_2,\mathbf{S}} = 1 & \qquad \|q\|_{s_2,\mathbf{S}} = 0\,. \end{array}$$

For the antecedent of φ we have $\|\neg\mathrm{S}(\neg p)\|^*_{\mathbf{S}} = 1$ because $\|p\|_{s_1,\mathbf{S}} = 1$. Furthermore, $\|p\|_{\mathbf{S}} = 0$ and $\|q\|_{\mathbf{S}} = u$. Therefore $\|\neg p\|^*_{\mathbf{S}} = 1$ and we get

$$\|\neg p \to q\|^*_{\mathbf{S}} = \|q\|_{\mathbf{S}} = u < 1\,.$$

Since $\|q\|_{\mathbf{S}} = u$ we get $\|\neg q\|^*_{\mathbf{S}} = 0$ and $\|\neg\neg q\|^*_{\mathbf{S}} = 1$. Thus, we get

$$\|\neg\neg q \to (\neg p \to q)\|^*_{\mathbf{S}} = \|\neg p \to q\|^*_{\mathbf{S}} < 1$$

for the succedent of φ. Therefore we get $\|\varphi\|^*_{\mathbf{S}} \neq 1$. □

Theorem 3.4. *If $*$ is a continuous t-norm that is not isomorphic to the Łukasiewicz t-norm, then g-validity and p-validity are not equivalent in* S$*$.

3.3. Equivalence of p-validity and u-validity in $\mathrm{S}\mathsf{P}$

For proving the equivalence of p-validity and u-validity under the Product t-norm we have to adapt the proof that we gave for the Łukasiewicz t-norm. The crucial observation there was that the Łukasiewicz residuum is continuous. This is not the case with the Product residuum which is given by

$$x \Rightarrow_{\mathsf{P}} y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$$

and is not continuous at the point $(0, 0)$. Consider for example $\lim_{x\to 0}(x{\Rightarrow_{\mathsf{P}}}0) = 0$ vs. $(0{\Rightarrow_{\mathsf{P}}}0) = 1$. However, $(0, 0)$ is the only discontinuity of the Product residuum and therefore we can overcome this issue by being especially careful in dealing with subformulas that have the truth value 0. In the following lemma we observe a condition under which two precisification spaces agree on formulas having the truth value 0. We then use the lemma to prove the equivalence.

Lemma 3.5. *Let φ be a formula and $\mathbf{S}$ and $\mathbf{S}'$ precisification spaces that fulfill the following conditions:*

- $\|p\|_{\mathbf{S}} = 0$ *if and only if* $\|p\|_{\mathbf{S}'} = 0$ *for every propositional variable p.*
- $\|\mathrm{S}\psi\|_{\mathbf{S}} = \|\mathrm{S}\psi\|_{\mathbf{S}'}$ *for every subformula* $\mathrm{S}\psi$ *of φ.*

Then we have $\|\varphi\|^{\mathsf{P}}_{\mathbf{S}} = 0$ *if and only if* $\|\varphi\|^{\mathsf{P}}_{\mathbf{S}'} = 0$.

Proof. The proof is by induction on the complexity of φ:

- $\varphi = \bar{0}$: Clear.
- $\varphi = \mathrm{S}\psi$ or $\varphi = p$ (for propositional p): By assumption.
- $\varphi = \psi \,\&\, \chi$: The truth function of strong conjunction is multiplication. Therefore we get $\|\psi \,\&\, \chi\|^{\mathsf{P}}_{\mathbf{S}} = 0$ if and only if $\|\psi\|^{\mathsf{P}}_{\mathbf{S}} = 0$ or $\|\chi\|^{\mathsf{P}}_{\mathbf{S}} = 0$. By the induction hypothesis this condition is equivalent to $\|\psi\|^{\mathsf{P}}_{\mathbf{S'}} = 0$ or $\|\chi\|^{\mathsf{P}}_{\mathbf{S'}} = 0$ which is equivalent to $\|\psi \,\&\, \chi\|^{\mathsf{P}}_{\mathbf{S'}} = 0$.
- $\varphi = \psi \to \chi$: By analyzing the residuum $\Rightarrow_{\mathsf{P}}$ we notice that $(x \Rightarrow_{\mathsf{P}} y) = 0$ if and only if $y = 0$ and $x > 0$. Therefore we get the following chain of equivalences: $\|\psi \to \chi\|^{\mathsf{P}}_{\mathbf{S}} = 0$ if and only if $\|\psi\|^{\mathsf{P}}_{\mathbf{S}} \neq 0$ and $\|\chi\|^{\mathsf{P}}_{\mathbf{S}} = 0$ if and only if $\|\psi\|^{\mathsf{P}}_{\mathbf{S'}} \neq 0$ and $\|\chi\|^{\mathsf{P}}_{\mathbf{S'}} = 0$ if and only if $\|\psi \to \chi\|^{\mathsf{P}}_{\mathbf{S'}}$.

$\square$

Theorem 3.6. *For every formula φ, φ is u-valid in* SP *if and only if φ is p-valid in* SP*.*

Proof. Let φ^* be a formula that is u-valid in SP and let $\mathbf{S}$ be a positive precisification space with a probability measure μ and a finite number of precisifications $\mathbf{P} = \{s_1, \ldots, s_n\}$, which is sufficient due to Proposition 2.9. We will show that $\|\varphi^*\|^{\mathsf{P}}_{\mathbf{S}} = 1$. For every vector of real numbers $\vec{x} = (x_1, \ldots, x_n)$ such that $x_1 + \cdots + x_n = 1$ we define the precisification space $\mathbf{S}_{\vec{x}}$ as having the same set of precisifications $\mathbf{P}$ as $\mathbf{S}$ together with the same local truth values and a probability measure $\mu_{\vec{x}}$ that we define by $\mu_{\vec{x}}(s_i) = x_i$ for each $s_i \in \mathbf{P}$.

Let $\vec{x} \in (0,1]^n \cap \mathbb{Q}$ be a vector such that $\mathbf{S}_{\vec{x}}$ is a positive, rational precisification space. By the definition of $\mathbf{S}_{\vec{x}}$ we know the following:

- $\|p\|_{s_i, \mathbf{S}_{\vec{x}}} = \|p\|_{s_i, \mathbf{S}}$ for every propositional variable p and every $1 \leq i \leq n$.
- $\|\mathrm{S}\psi\|_{\mathbf{S}_{\vec{x}}} = \|\mathrm{S}\psi\|_{\mathbf{S}}$ for every formula ψ.

Next we show that $\|p\|_{\mathbf{S}} = 0$ if and only if $\|p\|_{\mathbf{S}_{\vec{x}}} = 0$ for every propositional variable p. By Lemma 3.5 this implies that $\|\varphi\|^{\mathsf{P}}_{\mathbf{S}} = 0$ if and only if $\|\varphi\|^{\mathsf{P}}_{\mathbf{S}_{\vec{x}}} = 0$ for every formula φ.

We already know that $\|\mathrm{S}\neg p\|_{\mathbf{S}} = 0$ if and only if $\|\mathrm{S}\neg p\|_{\mathbf{S}_{\vec{x}}} = 0$ for every propositional variable p. By Proposition 2.6, which may be applied in the case of positive precisification spaces, we get $\|\mathrm{S}\neg p\|_{\mathbf{S}} = 0$ if and only if $\|p\|_{\mathbf{S}} = 0$ as well as $\|\mathrm{S}\neg p\|_{\mathbf{S}_{\vec{x}}} = 0$ if and only if $\|p\|_{\mathbf{S}_{\vec{x}}} = 0$. Therefore we conclude $\|p\|_{\mathbf{S}} = 0$ if and only if $\|p\|_{\mathbf{S}_{\vec{x}}} = 0$.

It is now straightforward to check that the following recursive definition of the function f_φ fulfills $f_\varphi(\vec{x}) = \|\varphi\|^{\mathsf{P}}_{\mathbf{S}_{\vec{x}}}$ for every formula φ.

$$f_\varphi(\vec{x}) = \begin{cases} \|\mathrm{S}\psi\|_{\mathbf{S}} & \text{if } \varphi = \mathrm{S}\psi \\ \sum_{i=1}^{n} \|p\|_{s_i, \mathbf{S}} \cdot x_i & \text{if } \varphi = p \\ f_\psi(\vec{x}) *_{\mathsf{P}} f_\chi(\vec{x}) & \text{if } \varphi = \psi \,\&\, \chi \\ f_\psi(\vec{x}) \Rightarrow_{\mathsf{P}} f_\chi(\vec{x}) & \text{if } \varphi = \psi \to \chi,\ \|\psi\|^{\mathsf{P}}_{\mathbf{S}} \neq 0,\ \|\chi\|^{\mathsf{P}}_{\mathbf{S}} \neq 0 \\ f_\psi(\vec{x}) \Rightarrow_{\mathsf{P}} 0 & \text{if } \varphi = \psi \to \chi,\ \|\psi\|^{\mathsf{P}}_{\mathbf{S}} \neq 0,\ \|\chi\|^{\mathsf{P}}_{\mathbf{S}} = 0 \\ 1 & \text{if } \varphi = \psi \to \chi,\ \|\psi\|^{\mathsf{P}}_{\mathbf{S}} = 0 \end{cases}$$

Since we have fixed φ^* and $\mathbf{S}$, the function f_{φ^*} can be seen as a composition of the functions appearing in the recursive calls. We now want to show that the domains and ranges of these functions can be restricted to (products of) the half-open unit interval $(0, 1]$. Before we can prove this, we need the following claim.

Claim. *Let φ be a subformula of φ^* that fulfills the following conditions:*

- $\|\varphi\|^{\mathsf{P}}_{\mathbf{S}} = 0$
- *The occurrence of φ in φ^* is not in the scope of an* S*-operator.*

Then the following holds:

(1) φ^* has a subformula of the form $\psi \to \chi$ or of the form $\chi \to \psi$ such that φ is a subformula of ψ.

(2) There is such a subformula such that $\|\psi\|^{\mathsf{P}}_{\mathbf{S}} = 0$.

First, note that φ^* is u-valid and therefore $\|\varphi^*\|^{\mathsf{P}}_{\mathbf{S}_{\vec{x}}} = 1 \neq 0$ which, as we proved above, implies $\|\varphi^*\|^{\mathsf{P}}_{\mathbf{S}} \neq 0$. Suppose that part (1) of the claim is not true. In this case we know that either $\varphi^* = \varphi$ or that φ is of the form $\psi_1 \,\&\, \ldots \,\&\, \psi_k$ where $\varphi = \psi_i$ for some $1 \leq i \leq k$. Since $\|\varphi^*\|^{\mathsf{P}}_{\mathbf{S}} \neq 0$ both cases are not possible because $\|\varphi\|^{\mathsf{P}}_{\mathbf{S}} = 0$ and $x *_{\mathsf{P}} 0 = 0 *_{\mathsf{P}} x = 0$. Let $\psi \to \chi$ or $\chi \to \psi$, respectively be the innermost subformula of φ^* such that ψ contains φ, i.e., the one with minimal length. By repeating the argument from before we get $\|\psi\|^{\mathsf{P}}_{\mathbf{S}} = 0$ which completes the proof of the claim.

We know that the function f_{φ^*} can be restricted to the domain $(0, 1]^n$ because we are only interested in strictly positive values for $x_1, \ldots, x_n$ We now show, for every function in the recursive calls of f_{φ^*}, that if the domain of the function is restricted to $(0, 1]$, also the range is restricted to $(0, 1]$.

- If $x \in (0, 1]$ and $y \in (0, 1]$, then also $x *_{\mathsf{P}} y \in (0, 1]$.
- The case $\|\mathrm{S}\psi\|_{\mathbf{S}_{\vec{x}}} = 0$ cannot occur in the recursive calls of $f_{\varphi^*}(\vec{x})$ because in this case we would have $\|\mathrm{S}\psi\|_{\mathbf{S}} = 0$ which by our claim is already handled by one of the cases for implication.
- For the same reason, the case $\sum_{i=1}^{n} \|p\|_{s_i,\mathbf{S}} \cdot x_i = 0$ cannot occur in the recursive calls of $f_{\varphi^*}(\vec{x})$ because

  $$\sum_{i=1}^{n} \|p\|_{s_i,\mathbf{S}} \cdot x_i = \sum_{i=1}^{n} \|p\|_{s_i,\mathbf{S}_{\vec{x}}} \cdot x_i = \|p\|_{\mathbf{S}_{\vec{x}}}$$

  and $\|p\|_{\mathbf{S}_{\vec{x}}} = 0$ if and only if $\|p\|_{\mathbf{S}} = 0$.
- If $f_\psi(\vec{x}) > 0$ and $f_\chi(\vec{x}) > 0$, then also $f_\psi(\vec{x}) *_{\mathsf{P}} f_\chi(\vec{x}) > 0$ and $f_\psi(\vec{x}) \Rightarrow_{\mathsf{P}} f_\chi(\vec{x}) > 0$.
- If $f_\psi(\vec{x}) > 0$, then also $f_\psi(\vec{x}) \Rightarrow_{\mathsf{P}} 0 > 0$.

- In any case, $1 > 0$.

We have shown that f_{φ^*} is composed of functions that can be restricted to the interval $(0,1]$. All these functions are continuous in the interval $(0,1]$. Therefore f_{φ^*} is a continuous function $(0,1]^n \to (0,1]$. Just like in the proof of Theorem 3.1, we define the sequence of vectors $\vec{r}^{(j)}$ such that $\lim_{j\to\infty} r_i^{(j)} = \mu(i)$ and $r_i^{(j)} \in (0,1] \cap \mathbb{Q}$ for $1 \leq i \leq n$.[7] Since $\mu(i) \in (0,1]$ and $\vec{r}^{(j)} \in (0,1]^n$ for every $j \geq 0$ we get

$$\|\varphi^*\|^{\mathsf{P}}_{\mathbf{S}} = f_{\varphi^*}\left(\lim_{j\to\infty} \vec{r}^{(j)}\right) = \lim_{j\to\infty} f_{\varphi^*}\left(\vec{r}^{(j)}\right) = \lim_{j\to\infty} \|\varphi^*\|^{\mathsf{P}}_{\mathbf{S}_{\vec{r}^{(j)}}} = 1\,.$$

Since $\mathbf{S}$ was an arbitrary finite, positive precisification space we conclude that φ^* is p-valid. □

3.4. Equivalence of p-validity and u-validity in SG

In the following we give a prove that p-validity and u-validity are equivalent in SG. The key idea is that in Gödel logic only the order of the truth degrees is relevant, and not their exact values. In our setting, the truth values of propositional variables are given by sums of measures of precisifications. We show that every order on sums of measures that can be expressed with positive precisification spaces can also be expressed with uniform precisification spaces.

Lemma 3.7. *Let X be a system set of m linear equations and inequalities with n variables $x_1, \ldots, x_n$ of the form*

$$\sum_{i=1}^{n} a_{ij} \cdot x_i = 0 \text{ or } \sum_{i=1}^{n} a_{ij} \cdot x_i < 0$$

where each a_{ij} is a rational number, $1 \leq i \leq n$ and $1 \leq j \leq m$. Then X has a positive, rational solution if and only if X has a positive, real solution. If X has a positive, rational solution, then X also has a positive, rational solution such that the constraint $\sum_{i=1}^{n} a_{ij} \cdot x_i = 1$ is fulfilled.

Proof sketch. Note that we can eliminate inequalities by introducing (strictly positive) slack variables. The paper [24] describes an algorithm for finding a positive solution of a system X of linear equations if it has one. From the constructions of the algorithm it can be seen that this solution is rational if the coefficients of X are rational. The second part of the lemma follows straightforwardly by "normalizing" this solution. □

We can now construct from a precisification space with positive, real measures a second precisification space with positive, rational measures such that they are connected by certain conditions. We will subsequently show that these conditions are strong enough to determine the set of true formulas. For the rest of the proof, we introduce some simplifying notation.

[7] The construction guarantees that $\vec{r}^{(j)}$ gives a well-defined probability measure.

Definition 3.8. For a formula φ and a precisification space $\mathbf{S}$, the *extension* of φ is the set

$$[\varphi]_{\mathbf{S}} = \{s \in \mathbf{S} \mid \|\varphi\|_{s,\mathbf{S}} = 1\} .$$

Lemma 3.9. *Let $\mathbf{S}$ be a positive precisification space with a finite set of precisifications and let $\mathcal{P}$ be a set of propositional variables. Then there is a positive, rational precisification space $\mathbf{S}'$ such that the following conditions hold:*

(C1) $\|\mathrm{S}\varphi\|_{\mathbf{S}} = \|\mathrm{S}\varphi\|_{\mathbf{S}'}$ *for every formula* φ

(C2) $\|p\|_{\mathbf{S}} < \|q\|_{\mathbf{S}}$ *if and only if* $\|p\|_{\mathbf{S}'} < \|q\|_{\mathbf{S}'}$ *for all* $p, q \in \mathcal{P}$

(C3) $\|p\|_{\mathbf{S}} = 1$ *if and only if* $\|p\|_{\mathbf{S}'} = 1$ *for every* $p \in \mathcal{P}$

(C4) $\|p\|_{\mathbf{S}} = 0$ *if and only if* $\|p\|_{\mathbf{S}'} = 0$ *for every* $p \in \mathcal{P}$

Proof. Let $\mathbf{S}$ be a positive precisification space with a finite set of precisifications $\mathbf{P}$ and a probability measure μ. We consider a variable x_s for every $s \in \mathbf{P}$. For every propositional variable $p \in \mathcal{P}$ we define the linear combination L_p by

$$L_p = \sum_{s \in [p]_{\mathbf{S}}} x_s$$

where the sum of the empty set is 0. We define $x^*_s = \mu(s) > 0$ for every $s \in \mathbf{P}$. Note that, for every $p \in \mathcal{P}$ we have $\sum_{s\in[p]_{\mathbf{S}}} x^*_s = \|p\|_{\mathbf{S}}$. We define a system X of linear equations and inequalities as follows. For every pair $p, q \in \mathcal{P}$ we include the following equations or inequalities in X:

$$\begin{aligned} L_p &= L_q \text{ if } \|p\|_{\mathbf{S}} = \|q\|_{\mathbf{S}} \\ L_p &< L_q \text{ if } \|p\|_{\mathbf{S}} < \|q\|_{\mathbf{S}} \\ L_q &< L_p \text{ if } \|p\|_{\mathbf{S}} > \|q\|_{\mathbf{S}} . \end{aligned}$$

It is clear that, by subtracting the right hand sides, the system X is equivalent to a system X' that fulfills the precondition of Lemma 3.7. Note that in X' only rational coefficients appear. Since $(x^*_s)_{s\in\mathbf{P}}$ is a positive, real solution of X' we know by Lemma 3.7 that there exists a positive, rational solution $(x'_s)_{s\in\mathbf{P}}$ of X' such that $\sum_{s\in\mathbf{P}} x'_s = 1$.

We define a precisification space $\mathbf{S}'$ that is just like $\mathbf{S}$ but with a different probability measure. This means that the set of precisifications of $\mathbf{S}'$ is $\mathbf{P}$ and the local truth value of a propositional variable p, for every $s \in \mathbf{P}$ is $\|p\|_{s,\mathbf{S}'} = \|p\|_{s,\mathbf{S}}$. We define the probability measure μ' of $\mathbf{S}'$ by setting $\mu'(s) = x'_s$ for every $s \in \mathbf{P}$. Then μ' is well-defined because

$$\mu'(\mathbf{P}) = \sum_{s\in\mathbf{P}} \mu'(s) = \sum_{s\in\mathbf{P}} x'_s = 1 .$$

Furthermore $\mathbf{S}'$ is a positive precisification space because, for every $s \in \mathbf{P}$, $\mu'(s) = x'_s > 0$ since $(x'_s)_{s\in\mathbf{P}}$ is a positive solution. We now show that $\mathbf{S}'$ has the desired properties.

Since $\mathbf{S}$ and $\mathbf{S}'$ have the same sets of precisifications with the same sets of truth values assigned to them we have $[\varphi]_{\mathbf{S}} = [\varphi]_{\mathbf{S}'}$ for every formula φ. Therefore the following equivalences hold for every formula φ:

$$\begin{aligned} \|\mathrm{S}\varphi\|_{\mathbf{S}} &= 1 \text{ if and only if} \\ [\varphi]_{\mathbf{S}} &= \mathbf{P} \text{ if and only if} \\ [\varphi]_{\mathbf{S}'} &= \mathbf{P} \text{ if and only if} \\ \|\mathrm{S}\varphi\|_{\mathbf{S}'} &= 1 \end{aligned}$$

Since, for every formula φ, $\|\mathrm{S}\varphi\|_{\mathbf{S}} \in \{0,1\}$ and $\|\mathrm{S}\varphi\|_{\mathbf{S}'} \in \{0,1\}$ we may conclude $\|\mathrm{S}\varphi\|_{\mathbf{S}} = \|\mathrm{S}\varphi\|_{\mathbf{S}'}$ which proves (C1).

To prove (C3) and (C4) we apply Proposition 2.6 and get the equivalences

$$\begin{aligned} \|p\|_{\mathbf{S}} &= 1 \text{ if and only if} \\ \|\mathrm{S}p\|_{\mathbf{S}} &= 1 \text{ if and only if} \\ \|\mathrm{S}p\|_{\mathbf{S}'} &= 1 \text{ if and only if} \\ \|p\|_{\mathbf{S}'} &= 1 \end{aligned}$$

and

$$\begin{aligned} \|p\|_{\mathbf{S}} &= 0 \text{ if and only if} \\ \|\mathrm{S}\neg p\|_{\mathbf{S}} &= 1 \text{ if and only if} \\ \|\mathrm{S}\neg p\|_{\mathbf{S}'} &= 1 \text{ if and only if} \\ \|p\|_{\mathbf{S}'} &= 0 \end{aligned}$$

Since $[\varphi]_{\mathbf{S}} = [\varphi]_{\mathbf{S}'}$ for every formula φ, we in particular have $[p]_{\mathbf{S}} = [p]_{\mathbf{S}'}$ for every $p \in \mathcal{P}$. Therefore we get, for every $p \in \mathcal{P}$,

$$\|p\|_{\mathbf{S}'} = \sum_{s \in [p]_{\mathbf{S}'}} \mu'(s) = \sum_{s \in [p]_{\mathbf{S}}} \mu'(s) = \sum_{s \in [p]_{\mathbf{S}}} x'_s \,.$$

For the proof of (C2) assume first that $\|p\|_{\mathbf{S}} < \|q\|_{\mathbf{S}}$. Since $\|p\|_{\mathbf{S}} = \sum_{s \in [p]_{\mathbf{S}}} x^*_s$ and $\|q\|_{\mathbf{S}} = \sum_{s \in [q]_{\mathbf{S}}} x^*_s$ we have $\sum_{s \in [p]_{\mathbf{S}}} x^*_s < \sum_{s \in [q]_{\mathbf{S}}} x^*_s$. Since an inequality that is equivalent to $L_p < L_q$ is contained in X', we get $\sum_{s \in [p]_{\mathbf{S}}} x'_s < \sum_{s \in [q]_{\mathbf{S}}} x'_s$. Since $\|p\|_{\mathbf{S}'} = \sum_{s \in [p]_{\mathbf{S}}} x'_s$ and $\|q\|_{\mathbf{S}'} = \sum_{s \in [q]_{\mathbf{S}}} x'_s$, we conclude $\|p\|_{\mathbf{S}'} < \|q\|_{\mathbf{S}'}$.

Now assume that $\|p\|_{\mathbf{S}} \not< \|q\|_{\mathbf{S}}$. If $\|p\|_{\mathbf{S}} > \|q\|_{\mathbf{S}}$, then the same reasoning as before applies and we get $\|p\|_{\mathbf{S}'} > \|q\|_{\mathbf{S}'}$. If $\|p\|_{\mathbf{S}} = \|q\|_{\mathbf{S}}$, then also a similar argument gives us $\|p\|_{\mathbf{S}'} = \|q\|_{\mathbf{S}'}$. In both cases we have $\|p\|_{\mathbf{S}'} \not< \|q\|_{\mathbf{S}'}$. □

We now show that the conditions of the previous lemma are sufficient for two precisification spaces to have the same sets of true formulas.

Lemma 3.10. *Let φ be a formula, let $\mathcal{P}$ be the set of propositional variables of φ, and let $\mathbf{S}$ and $\mathbf{S}'$ be precisification spaces such that conditions (C1)–(C4) hold. Then $\|\varphi\|^{\mathsf{G}}_{\mathbf{S}} = 1$ if and only if $\|\varphi\|^{\mathsf{G}}_{\mathbf{S}'} = 1$.*

Proof sketch. The proof is straightforward by induction on the complexity of φ.[8] It is convenient to include in the induction hypothesis the fact that $\|\varphi\|^{\mathsf{G}}_{\mathbf{S}} = \|p\|^{\mathsf{G}}_{\mathbf{S}}$ if and only if $\|\varphi\|^{\mathsf{G}}_{\mathbf{S}'} = \|p\|^{\mathsf{G}}_{\mathbf{S}'}$ for every propositional variable $p \in \mathcal{P}$ (see also Proposition 4.4 of [25]). □

Theorem 3.11. *For every formula φ, φ is u-valid in* SG *if and only if φ is p-valid in* SG.

Proof. Assume that φ is u-valid and let $\mathbf{S}$ be a positive precisification space with a set of precisifications $\mathbf{P}$ and a probability measure μ. We assume that $\mathbf{P}$ is finite which is sufficient due to Proposition 2.9. By Lemma 3.9 there is a precisification space $\mathbf{S}'$ with a probability measure μ' such that $\mu'(s) \in \mathbb{Q}^{>0}$ for every $s \in \mathbf{P}'$ and conditions (C1)–(C4) hold. By Lemma 3.10 we then know that $\|\varphi\|^{\mathsf{G}}_{\mathbf{S}} = 1$ if and only if $\|\varphi\|^{\mathsf{G}}_{\mathbf{S}'} = 1$. Since φ is u-valid we know that $\|\varphi\|^{\mathsf{G}}_{\mathbf{S}'} = 1$. Therefore we get $\|\varphi\|^{\mathsf{G}}_{\mathbf{S}} = 1$. As $\mathbf{S}$ was an arbitrary finite, positive precisification space we conclude that φ is p-valid. □

4. Conclusion

We have characterized the t-norms for which g-validity and p-validity are equivalent. Furthermore, we showed that p-validity and u-validity are equivalent for the three most important t-norms: the Łukasiewicz, the Gödel, and the Product t-norm. The continuity of the Łukasiewicz t-norm ensures a certain robustness which has also been observed in other contexts. With exception to the Łukasiewicz t-norm, we observed a gap in g-validity and p-validity which means that the most important design choice for a hybrid logic is whether 0-measured precisifications should be allowed in a precisification space. Compared to this, the gap in p-validity and u-validity is smaller, if not non-existent at all. Therefore the question whether precisifications should be measured uniformly seems to be less crucial.

Characterizing the t-norms for which the equivalence between p-validity and u-validity holds remains an open problem. For S-free formulas this is equivalent to asking for which t-norms the real and the rational semantics coincide in terms of validity. To the best of our knowledge, this problem seems to be open, except for the three t-norms already considered in this article. We hope to have provided some further motivation for this problem.

The proof theory of our hybrid logics has not been studied very well. A tableaux-style proof system for the logic SŁ, which is based on a game-theoretic interpretation, has been given by Fermüller and Kosik [4]. Open problems include axiomatizations and Gentzen-style proof systems for the hybrid logics.

[8] Note that the Gödel t-norm is given by $x *_{\mathsf{G}} y = \min(x, y)$ and its residuum is given by $x \Rightarrow_{\mathsf{G}} y = 1$ if $x \leq y$ and $x \Rightarrow_{\mathsf{G}} y = y$ if $x > y$.

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