# Limiting Price Discrimination when Selling Products with Positive Network Externalities*

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**Abstract.** Assume a seller wants to sell a *digital product* in a *social network* where a buyer's valuation of the item has positive network externalities from her neighbors that already have the item. The goal of the seller is to maximize his revenue. Previous work on this problem [9] studies the case where clients are offered the item in sequence and have to pay personalized prices. This is highly infeasible in large scale networks such as the Facebook graph: (1) Offering items to the clients one after the other consumes a large amount of time, and (2) *price-discrimination* of clients could appear unfair to them and result in negative client reaction or could conflict with legal requirements.

We study a setting dealing with these issues. Specifically, the item is offered in parallel to multiple clients at the same time and at the same price. This is called a round. We show that with $O(\log n)$ rounds, where $n$ is the number of clients, a constant factor of the revenue with price discrimination can be achieved and that this is not possible with $o(\log n)$ rounds. Moreover we show that it is APX-hard to maximize the revenue and we give constant factor approximation algorithms for various further settings of limited price discrimination.

## 1 Introduction

With the appearance of *online social networks* the issue of monetizing network information arises. Many digital products such as music, movies, apps, e-books, and computer games are sold via platforms with social network functionality. Often these products have so called *positive network externalities*: the valuation of a client for a product increases (potentially marginal) when a related client (e.g., a friend) buys the product, i.e., a product appears more valuable for a client if a friend already owns the same product. In the presence of positive network externalities, motivating a client to buy a product by lowering the price he has to pay could incentivize his friends to also buy the product. Consequently, it could increase future revenue. Thus, when trying to maximize revenue there is an interesting trade off between the current and the future revenue.

We follow the work of Hartline et al. [9] for modeling network externalities and (marketing) strategies in social networks with the goal of maximizing the

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seller's revenue. We model the seller's information about the clients' valuations by using a directed weighted graph or, more generally, by using submodular set functions. For our positive results we assume that the seller has only incomplete information about the valuations, i.e., we are in a Bayesian setting where the seller is given a distribution of each client's valuation but not the valuation itself. The seller's *strategy* decides (a) when to offer the product to a client and (b) at which price. Thus each strategy assigns each client a (time, price)-pair.

In [9] a setting with *full price discrimination* is studied where a seller offers a product sequentially for a different price to different clients. (See [2, 6, 7] for further work on settings with full price discrimination.) While it gives a good baseline to compare with, this approach has multiple drawbacks. First, processing one client after the other and waiting for the earlier client's decision requires too much time if the number of clients is large. Second, price-discrimination could appear unfair to the clients and could result in a negative reaction of some clients [12]; moreover, it might be in conflict with legal requirements. On the other side offering all clients the *same* price (i.e., a *uniform price*) reduces the revenue significantly, namely, by a factor of $\log n$, as we show below.

Hence, we introduce *rounds* such that in a round the product is offered to a set of clients at the *same* time and we consider strategies with a limited number of rounds and/or limited price discrimination. Specifically we study $k$-round strategies where the product is offered to the clients in $k$ rounds with $1 \leq k \leq n$ such that only clients that have purchased the product in a *previous round* can influence the valuation functions of the clients to whom the product is offered in the current round. Following [9] the strategies offer the product only once to each client to avoid that clients behave strategically, i.e., wait for price decreases in the future. We study two types of $k$-round strategies: (1) *$k$-$\mathcal{PD}$ strategies* where each client has a personalized price (*limited number of rounds*), and (2) *$k$-$\mathcal{PR}$ strategies* where all clients in the same round are offered the same price (*limited number of rounds and limited price discrimination*). Thus, $k$-$\mathcal{PR}$ strategies generalize the simple uniform price setting, where one might distribute free copies at the beginning and then charge everyone the same price. The setting in [9] corresponds to the $n$-$\mathcal{PR}$ setting. Throughout the paper we use $\widehat{R}$ to denote the optimal revenue achievable by an $n$-$\mathcal{PR}$ strategy. To summarize we study the natural question of how many different prices/rounds are necessary and sufficient if we want to achieve a constant factor approximation algorithm.

Our main results are: (1) There is a $\log n$-$\mathcal{PR}$ strategy that achieves a constant factor approximation of $\widehat{R}$ for very general valuation functions, namely probabilistic submodular valuation functions. Thus only $\log n$ different prices are necessary. We show that this result is tight (up to constant factors) in two regards: (a) It cannot be achieved with $o(\log n)$ rounds, even for very limited valuations with deterministic and additive externalities. (b) There exists a constant $c$ such that it is NP-hard to compute a $c$-approximation of $\widehat{R}$, no matter how many rounds, i.e., maximizing revenue in $k$-$\mathcal{PR}$ strategies (as well as $k$-$\mathcal{PD}$ strategies) is APX-hard. (2) There is a 2-$\mathcal{PR}$ strategy that achieves an $O(\log n)$-approximation of $\widehat{R}$. (3) We give (nearly) 1/16-approximation algorithms for the maximum revenue achievable by any $k$-$\mathcal{PR}$ strategy when compared to the optimal $k$-$\mathcal{PR}$ strategy (i.e., not compared with $\widehat{R}$).

All algorithms we present are polynomial in the number of clients. Interestingly, all of them make very limited but also very natural use of the network structure: They only exploit information about the neighbors of a node and not any global properties.

*Discussion of Related work.* In the presence of network externalities two main types of revenue maximizing strategies have been used in the literature: strategies *with price discrimination* (each client pays a different price), and strategies *with uniform price* (every paying client pays the same price). We have already mentioned the work of Hartline et al. [9] on price discriminating strategies with $n$ rounds ($n$-$\mathcal{PD}$). They study so-called *influence-and-exploit* strategies consisting of two steps: (1) the *influence* step, in which the product is given for free to a set of *influence nodes;* and (2) the *exploit* step, in which clients are approached sequentially and each client is offered the product at a personalized price. Hartline et al. give a randomized influence-and-exploit strategy that gives an $\frac{e}{4e-2}$-approximation of the optimal revenue $\widehat{R}$. Note that the revenue of our generalization to $k$-$\mathcal{PD}$ in Section 3 is quite close to this as it gives an $\frac{e}{4e-2+2/(k-1)}$-approximation. In particular for $k \geq 10$ our strategy $k$-PD$(q)$ achieves more than 95% of the revenue of their strategy. Moreover with the improvements of Theorem 3 and $k \geq 10$ we get even better constants than [9]. Additionally, Hartline et al. present an algorithm that, together with a novel result on submodular function maximization [4], 0.5-approximates the optimal influence set.

Later, Fotakis and Siminelakis [7] studied a restricted model of client valuations and improved the approximation algorithms of Hartline et al. [9]. Furthermore, they study the ratio between the optimal strategies and the optimal influence-and-exploit strategies. Babaei et al. [2] experimentally evaluated several marketing strategies without an influence step, instead giving the most influential clients *discounts*. They conclude that discounts increase the revenue in the considered artificial and real networks.

Influence-and-exploit strategies with uniform prices have been studied by Mirrokni et al. [11]. They use generic algorithms for submodular function maximization to obtain an influence set with at least $1/2$ of the revenue of the optimal uniform price influence-and-exploit strategy, *not* of $\widehat{R}$. In both, [9] and [11], similar *graph models with concave influences* (CG) are introduced as model for network externalities.

Akhlaghpour et al. [1] study a different scenario with uniform prices, without any price discrimination: The product is offered on $k$ consecutive rounds to all clients for the same price, and a client buys the product when its value exceeds the price for the day. They present an FPTAS for the *Basic* scenario, where a client buying the product immediately influences the valuation of the other clients who then may also buy the product in the same round. A round ends (the price changes) when no client is willing to buy the product for the current price. This model does not fit well to our assumptions of limited time and a large networks. In the *Rapid* scenario, where buyers on the same day do not affect each other (like in our setting) they show that no constant factor approximation is possible and give an $O(\log_k n)$ approximation. The main difference to our setting is that in [1] the product has to be offered to every client (not having the product) in each round and thus there is no influence round where clients get the product for free.

For non-digital goods Ehsani et al. [6] study revenue maximizing strategies with both price discrimination and uniform prices with full information about the clients valuations and with *production costs* per unit in a setting where clients arrive randomly. They give an FPTAS for the optimal uniform price. Recall that we are studying digital goods without full information about the valuation functions.

Haghpanah et al. [8] study submodular network externalities for bidders in auctions and provide auctions that give a 0.25-approximation of the optimal revenue. In our models the strategies have to offer items in rounds since the clients are only influenced by clients, who bought the product in a previous round; that is an important difference to the auctions in [8].

*Structure of the paper.* In Section 2, we present our model for networks externalities as well as the different kinds of marketing strategies we consider in this paper. In Section 3, we study the effect of restricting the number of rounds, i.e., the effect of offering the product to several clients in parallel, but allowing full price discrimination. In Section 4 we compare the optimal revenue achievable with individual prices against the optimal revenue achievable with uniform prices. Efficiently computable $k$-$\mathcal{PR}$ strategies are studied in Section 5. In Section 6 we discuss several extensions of our model. Finally, in the Appendix we provide all omitted proofs.

## 2 Preliminaries

We are given a network $G = (V, E)$ of $n$ clients $V$ and edges $E \subseteq V \times V$ that represent their relationships. Suppose that we want to offer a digital product (i.e., the unit costs of the product are zero and we can produce an arbitrary number of copies) to each client $i \in V$ for some price $p_i \in \mathbb{R}_{\geq 0}$ and maximize our revenue. We call a client that has bought our product *active client* or *buyer*; otherwise we call him *inactive client*. We define the valuation of client $i \in V$ by $v_i : V \setminus \{i\} \to \mathbb{R}_{\geq 0}$, such that the valuation of $i$ only depends on his neighbors, i.e., for $A$ being the set of active clients and $N_i$ being the set of neighbors of $i$ in $G$ holds $v_i(A) = v_i(A \cap N_i)$. We try to exploit this dependency of the valuation on the status of the neighbors (called *externality*) by offering the product to clients in a certain order, i.e., we want to compute an order on the clients. Furthermore, we restrict ourselves to a single offer to each client. We assume that clients are *individually rational* and have *quasi-linear utilities*. Thus client $i$ buys the product if and only if the price $p_i$ is not larger than the valuation $v_i(A)$.

**Valuation functions.** We describe the different models for externalities for a client $i \in V$ and his active neighbors $B$. Our main focus is on submodular[1] valuation functions, based on the intuition that the positive influence of a fixed neighbor does not increase when the set of active neighbors grows. Next we define the models we consider in the paper:

– *Simple Additive Model (SA).* There are non-negative weights $w_{i,i}$ and $w_{i,j}$ for $(i,j) \in E$. The valuation of $i$ is given by $v_i(B) = \sum_{j \in B \cup \{i\}} w_{i,j}$.
– *Deterministic Submodular Model (DS).* The valuation of $i$ is given by $v_i(B) = g_i(B)$, where $g_i : V \setminus \{i\} \to \mathbb{R}_{\geq 0}$ is a monotone, submodular set function.
– *Probabilistic Submodular Model (SM) [8].* The valuation of $i$ is given by $v_i(B) = \tilde{v}_i \cdot g_i(B)$ where $g_i : V \setminus \{i\} \to [0,1]$ is a (publicly known) monotone, submodular function with $g_i(V \setminus \{i\}) = 1$ and the private value $\tilde{v}_i \geq 0$ is drawn from a (publicly known) distribution with the CDF $F_i$.

In the first two models the seller has full information about the valuation while in the SM-model she only knows the distribution. We have that the DS-model generalizes the SA model and the SM-model generalizes both the SA and the DS-model. To simplify the presentation in this paper we state positive results for the SM-model and hardness results (whenever possible) for the SA-model. For all the models we call $v_i(\emptyset)$ the intrinsic valuation of client $i \in V$. Note that $v_i(V \setminus \{i\})$ is the maximum valuation of client $i \in V$ in each model. We will use this fact for upper bounds on the revenue any strategy can extract from a client.

**Seller information.** By the previous definitions the valuation functions model the information of the seller about the real valuation of the clients. If the seller has full information, the seller maximizes her revenue by setting $p_i = v_i(B)$, where $B$ are the active neighbors of $i \in V$. For the case of incomplete information price setting is more challenging. In particular, multiple prices could maximize the expected payment, the so-called myopic prices, and we do not know in general how likely it is that the client accepts one of those myopic prices.

**Definition 1.** *Given a client $i \in V$, and the set of active clients $B \subseteq V \setminus \{i\}$, the* myopic prices *of client $i$ are defined as* $\text{argmax}_{p \in \mathbb{R}_{\geq 0}} p \cdot \mathbf{P}[v_i(B) \geq p]$. *If $\mathbf{P}[v_i(B) \geq 0] = 0$ we define zero as the unique myopic price.*

The frequently used monotone hazard rate condition implies that there is a unique myopic price. The *hazard rate* of a PDF $f_{i,B}$ is defined to be $\frac{f_{i,B}(y)}{1-F_{i,B}(y)}$ for $y \geq 0$. If the hazard rate of $f_{i,B}$ is a monotone non-decreasing function of $y \geq 0$ where $y$ satisfies $F_{i,B}(y) < 1$ we say that $F_{i,B}$ has a *monotone hazard rate*. In the *SM-model with monotone hazard rates* we assume that for each $i \in V$ the CDF $F_i$ has monotone hazard rate. Note that for full information models, $F_{i,B}(y) = 0$ and $f_{i,B}(y) = 0$ for all $y < v_i(B)$ and $F_{i,B}(y) = 1$ for all $y \geq v_i(B)$.

**Lemma 1 ([9]).** *In the SM-model with monotone hazard rates each client has a unique myopic price which is accepted with probability at least $\frac{1}{e}$. For a set of active clients $B$ we denote the unique myopic price of client $i$ as $\hat{p}_i(B)$.*

[1] A set function $f : 2^S \to \mathbb{R}$ is called submodular if for all $X \subseteq Y \subseteq S$ and each $x \in S \setminus Y$ it holds that $f(X \cup \{x\}) - f(X) \geq f(Y \cup \{x\}) - f(Y)$.

We will also assume that we can compute this myopic price in polynomial time.

**(Marketing) Strategies.** A marketing strategy determines in each round to which clients and at what prices the product is offered in this round. This choice may depend on the already visited clients, the active clients and the number of rounds remaining.

**Definition 2.** *A* $k$-round (marketing) strategy *is a probabilistic function* $s$ *that maps* $(C, B, j)$ *to* $(V_j, p)$*, where* $C \subseteq V$ *are the clients visited so far,* $B \subseteq C$ *are the buyers in the previous rounds,* $j \in [k]$ *is the current round,* $V_j \subseteq V \setminus C$ *are the clients in round* $j$*, and* $p\colon V_j \to \mathbb{R}_{\geq 0}$ *gives the prices for clients in* $V_j$*.*[2]

In the following we consider two classes of $k$-round strategies, (i) one where each client gets an individual price, i.e., the rounds only influence the valuations and place no additional restrictions on the price, and (ii) one where the seller must set uniform prices in each round, and for each of them also the subclass of *Influence and Exploit (IE)*-strategies, where the seller offers the product for free to the clients of the first round, i.e., the price in the first round is fixed to 0. In 1-round strategies the seller has to offer the product to all clients at the same time and thus network externalities do not come into play at all.

**Price discrimination.** [$k$-$\mathcal{PD}$] This class contains all $k$-round strategies. If we set $k = |V|$ we get the class of all possible strategies. For $k \geq 2$ we consider the subclass [$k$-$\mathcal{PD}^{\text{IE}}$] of *$k$-round influence and exploit strategies*, where the seller gives the product for free to the clients selected in the first round.

**Uniform prices per round.** [$k$-$\mathcal{PR}$] This class contains all *$k$-round uniform price strategies*, where all clients visited in the same round are offered the same price. For $k \geq 2$ the *$k$-round uniform price influence and exploit strategies* [$k$-$\mathcal{PR}^{\text{IE}}$] are the $k$-$\mathcal{PR}$ strategies with the first round having uniform price 0.

Given a strategy $s$ we use $R(s)$ to denote the (expected) revenue obtained by $s$. By $\widehat{R}_{k\text{-}\mathcal{PD}}$, $\widehat{R}^{\text{IE}}_{k\text{-}\mathcal{PD}}$, $\widehat{R}_{k\text{-}\mathcal{PR}}$ and $\widehat{R}^{\text{IE}}_{k\text{-}\mathcal{PR}}$ we denote the *optimal* revenues achievable by the above classes of strategies, where $\widehat{R} = \widehat{R}_{n\text{-}\mathcal{PD}}$ is the optimal revenue achievable by *any* strategy. Restricted to $k$-rounds and uniform prices, our main goals are to achieve constant factor approximations of $\widehat{R}$ and $\widehat{R}_{k\text{-}\mathcal{PR}}$.

## 3 Strategies with Individual Prices

In this section we analyze $k$-$\mathcal{PD}$ strategies, i.e., $k$-round strategies with full price discrimination. We first show that maximizing the revenue of such strategies is computationally hard even in the SA-model; in particular, we show that it is NP-hard to approximate better than within a factor of $34k/(1+34k)$. The reduction uses the fact that even in the SA-model, the problem of maximizing the revenue of $k$-$\mathcal{PD}$ generalizes Maximum $k$-Cut, and the latter is APX-hard [10].

**Theorem 1.** *Maximizing the revenue of $k$-$\mathcal{PD}$, resp. $k$-$\mathcal{PD}^{IE}$, is* APX*-hard in the SA-model; it is* NP*-hard to* $\frac{34k}{1+34k}$*-approximate* $\widehat{R}_{k\text{-}\mathcal{PD}}$*, resp.* $\widehat{R}^{IE}_{k\text{-}\mathcal{PD}}$*, for* $k \geq 2$*.*

[2] In principle it suffices that $s(\cdot, \cdot, j)$ is defined on the possible outcomes of round $j-1$. In particular for $j = 1$ it has only to be defined for $s(\emptyset, \emptyset, 1)$.

On the positive side we generalize the result in [9] to the SM-model with monotone hazard rates and to $k$ rounds. Specifically, we show that the following IE strategy gives a constant factor approximation of the optimal $k$-round revenue as well as of the optimal revenue $\widehat{R}$ and can be computed in polynomial time. We will use these results in the following sections as they imply that any $k$-$\mathcal{PR}$ strategy that is an $\alpha$-approximation of $\widehat{R}_{k\text{-}\mathcal{PD}}$ is an $O(\alpha)$-approximation of $\widehat{R}$.

**Algorithm 1 (Strategy $k$-PD$(q)$).** Let $q$ be in $[0,1]$.

1. Assign clients in $V$ independently with probability $q$ to set $V_1$. Give the clients in $V_1$ the product for free.
2. Partition the clients in $V \setminus V_1$ into sets $V_2, \ldots, V_k$ s.t. each client is in $V_j$ independently of the other client with probability $(1-q)/(k-1)$.
3. Offer the clients in $V_j$ the product in parallel for their myopic price.

To analyze the strategy we first consider the expected payment $\pi_i(S) = \hat{p}_i(S) \cdot \mathbf{P}\left[v_i(S) \geq \hat{p}_i(S)\right]$ we can extract from a client $i$ given the active clients $S$. By the definition of the myopic price, we can show that $\hat{p}_i(S) = \hat{p}_i(V \setminus \{i\}) \cdot g_i(S)$. The crucial idea in Lemma 2 is now that we can lower bound the expected revenue $\pi_i(S)$ collected from client $i$ by the maximum revenue that can be collect from $i$, namely $\pi_i(V \setminus \{i\})$, multiplied by the probability $\beta$ that a client is in $S$. Note that $\beta$ is a function of $q$. Theorem 2 then determines the value $\beta$, which in turn sets $q$. Finally we use a well-known property of submodular functions [9] to lower bound the revenue of $k$-PD$(q)$.

**Lemma 2.** *Let $S \subseteq V \setminus \{i\}$ be the random set of clients and let each client $j \in V \setminus \{i\}$ be in $S$ independently with a probability of at least $\beta$. Then it holds that $\mathbf{E}_S\left[\pi_i(S)\right] \geq \beta \cdot \pi_i(V \setminus \{i\})$.*

**Theorem 2.** *Consider the SM-model with monotone hazard rates. For $q = 1 - \frac{e\cdot(k-1)}{2e(k-1)-k+2}$ it follows that $R(k\text{-PD}(q)) \geq \frac{e\cdot(k-1)}{4e(k-1)-2k+4}\widehat{R}$ and thus also $\widehat{R}_{k\text{-}\mathcal{PD}} \geq \frac{e\cdot(k-1)}{4e(k-1)-2k+4}\widehat{R}$.*

For $k=2$, buyers are only influenced by clients in the influence set and Theorem 2 gives a 2-$\mathcal{PD}$ strategy, i.e., 2-PD(0.5), which achieves at least $\frac{1}{4}$ of the optimal revenue (a similar result was given in [9]). However the main challenge in the above algorithm is to exploit also the externalities from the other preceding rounds. The 2-round case will be crucial for our $k$-$\mathcal{PR}$-algorithm in Section 5.

**Corollary 1.** *Given the SM-model with monotone hazard rates, it follows that $\widehat{R}_{k\text{-}\mathcal{PD}} \geq \widehat{R}^{IE}_{k\text{-}\mathcal{PD}} \geq \frac{1}{4} \cdot \widehat{R}$ for $k \geq 2$.*

Fotakis and Siminelakis [7] show that myopic prices are not necessarily optimal for IE-strategies. They provide IE-strategies, using lower prices, that beat those of [9] if the valuations follow the uniform additive model. In the following we generalize this idea to (a) submodular valuations with monotone hazard rates and (b) to the $k$-round setting. That is, we consider strategies that use different discount factors $\alpha_j$ for different rounds $j$. To be more precise in each round the seller charges every client only an $\alpha_j$-fraction of his myopic price. Moreover, we also consider different probabilities $q_j$ for a client being assigned to round $j$. The next theorem shows that charging less than the myopic price can improve the overall revenue.

**Algorithm 2 (Strategy** $k$-PD$(\bar{q}, \bar{\alpha})$**).** Let $\bar{q}$ and $\bar{\alpha}$ be vectors of length $k$ with entries in $[0, 1]$ and let the entries of $\bar{q}$ sum up to 1.

1. Partition $V$ into sets $V_1, \ldots, V_k$ s.t. each client $i \in V$ is in $V_j$ independently of the others with probability $q_j$.
2. Offer the clients $i \in V_j$ the product in parallel for price $\alpha_j \cdot \hat{p}_i$ where $\hat{p}_i$ is the myopic price of client $i$.

**Theorem 3.** *Given the SM-Model with monotone hazard rates, for each $k$ there exist vectors $\bar{q}, \bar{\alpha}$ such that $R(k\text{-PD}(\bar{q}, \bar{\alpha})) \geq C_k \cdot \widehat{R}$, where $C_3 = 0.279$, $C_5 = 0.298$, $C_8 = 0.308$, and $C_{10} = 0.311$.*

Computing $C_k$ is a multi-parameter optimization problem (with $2k$ parameters) that we solved numerically. More details are provided in the appendix. Finally, note that the Algorithms 1 and 2 use the network structure only in the computation of the myopic prices, which only requires to know the active neighbors.

## 4 Comparing Individual Prices to Uniform Prices

In this section we study $k$-$\mathcal{PR}$ strategies where there are $k$ rounds and in each round we offer the product to a subset of the clients for a uniform price. We first analyze the impact that restricting the strategies to be uniform price strategies has on the optimal revenue for a constant number of rounds. We show that in the SM-model the optimal revenue can decrease in the worst case by a factor of $\Theta(1/n)$. Thus, if we do not make assumptions on the probability distributions we cannot do better than in each round just selecting the most valuable client and offer the product to him for his myopic price. However, if we consider the SM-model with monotone hazard rates the optimal revenue can decrease in the worst case by a factor of $\Theta(1/\log n)$. As a result we will focus on models with monotone hazard rates in the remainder of the paper.

**Theorem 4.** *Assume that the valuations of the clients follow the SM-model.*

1. *For every $\varepsilon > 0$ and $k \geq 1$ there exists a network and valuations $v_i$ such that $\widehat{R}_{k\text{-}\mathcal{PR}} \leq \frac{k+\varepsilon}{n} \widehat{R}_{k\text{-}\mathcal{PD}}$.*
2. *For any network and valuations $v_i$, $\widehat{R}_{1\text{-}\mathcal{PR}} \geq \frac{1}{n} \widehat{R}_{1\text{-}\mathcal{PD}}$.*
3. *For any network and valuations $v_i$, $\widehat{R}^{IE}_{2\text{-}\mathcal{PR}} \geq \frac{1}{n} \widehat{R}^{IE}_{2\text{-}\mathcal{PD}} \geq \frac{1}{4n} \widehat{R}$.*

The next theorem shows three points: (1) Even with monotone hazard rates, no $k$-$\mathcal{PR}$ strategy can be better than a $k/H_n$-approximation of $\widehat{R}_{k\text{-}\mathcal{PD}}$ and, thus, also of $\widehat{R}$. Recall, that the SA-model satisfies the monotone hazard rate condition and is a special case of a DS and an SM-model and thus these negative results also extend to these models. Thus without price discrimination within a round no constant factor approximation of $\widehat{R}$ with $o(\log n)$ different rounds exists.

(2) Even with only 2 rounds the *optimal* 2-$\mathcal{PR}$ strategy achieves an $O(\log n)$ approximation of $\widehat{R}^{\mathrm{IE}}_{2\text{-}\mathcal{PD}}$, which, by Corollary 1, achieves a 4-approximation of $\widehat{R}$. Thus, the optimal 2-$\mathcal{PR}$ strategy achieves an $O(\log n)$ approximation of $\widehat{R}$. This is a large improvement over the negative result from Theorem 4, which holds for valuations that do not have monotone hazard rates. However, we show in the next section that computing the optimal 2-$\mathcal{PR}$ strategy is NP-hard.

**Theorem 5.** *Assume that valuations of the clients follow the SM-model with monotone hazard rates.*

1. *For each $k \geq 1$ there exists a network and valuations $v_i$ such that $\widehat{R}_{k\text{-}\mathcal{PR}} \leq \frac{k}{H_n} \widehat{R}_{k\text{-}\mathcal{PD}}$ (even in the SA-model).*
2. *For any network and valuations $v_i$, $\widehat{R}_{1\text{-}\mathcal{PR}} \geq \frac{1}{e \cdot H_n} \widehat{R}_{1\text{-}\mathcal{PD}}$.*
3. *For any network and valuations $v_i$, $\widehat{R}^{IE}_{2\text{-}\mathcal{PR}} \geq \frac{1}{e \cdot H_n} \widehat{R}^{IE}_{2\text{-}\mathcal{PD}}$.*

Recall that we want to achieve a constant approximation of $\widehat{R}$ using a uniform price strategy. Thus, in the next section we give a polynomial time computable $k$-$\mathcal{PR}$ strategy, with $k \in \Theta(\log n)$, that achieves a constant factor approximation.

## 5 Strategies with Uniform Prices

In the analysis of the algorithms in Section 3 we exploited that the *expected revenue* from a client $i$ was submodular in the set of active neighbors. This is not true in the $\mathcal{PR}$ setting, as we can only extract revenue from a client if his valuation is larger than the uniform price. Still we can show the following: (1) We give a polynomial-time approximation scheme (PTAS) for one round strategies in the SM-model with monotone hazard rates, i.e., for finding the optimal uniform price for one round. (2) We show that finding an optimal 2-$\mathcal{PR}$ strategy is not only NP-hard but also APX-hard, even for the $DS$-model. (3) We give a constant factor approximation of $\widehat{R}$ with $O(\log n)$ rounds for valuations from the SM-model with monotone hazard rates. (4) From Section 4 we know that $k$-$\mathcal{PR}$ strategies, where $k$ is a constant, lose a factor of $\log n$ of the optimum revenue when compared to $\widehat{R}$. Thus, in this case the best we can hope for is a constant factor approximation of $\widehat{R}_{k\text{-}\mathcal{PR}}$, *not* of $\widehat{R}$. We have two such results: (4a) We give a $(1/16 - \varepsilon)$ approximation of $\widehat{R}_{k\text{-}\mathcal{PR}}$ for the SM-model with monotone hazard rates. (4b) We show that under certain conditions we can even give a $(1/4 - \varepsilon)$-approximation of $\widehat{R}_{2\text{-}\mathcal{PR}}$. Combined with Theorem 5 this gives an $O(\log(n)/k)$-approximation of $\widehat{R}$ in $k$ rounds.

We first give a PTAS for computing the optimal price for the one round setting. This will be a useful tool for the 2-round setting.

**Algorithm 3.** Let $c = (1 - \varepsilon)^{-1}$ and $\varepsilon > 0$.

1. Compute $\hat{p}_{\max} = \max_{i \in V} \hat{p}_i$.
2. For all $j \in \{0, \ldots, \lfloor \log_c(e \cdot n) \rfloor\}$:
   Compute the expected revenue $R_j$ for the uniform price $p_j = \frac{\hat{p}_{\max}}{c^j}$.
3. Return $p_j$ with maximal $R_j$.

**Theorem 6.** *Given the SM-model with monotone hazard rates, then for each* $\varepsilon > 0$ *Algorithm 3 gives a* $1\text{-}\mathcal{PR}$ *strategy* $s$ *(i.e., a uniform price), such that* $R(s) \geq (1 - \varepsilon) \cdot \widehat{R}_{1\text{-}\mathcal{PR}}$ *in polynomial time.*

The basic idea of the proof is that the optimal uniform price $p^*$ cannot be less than $\hat{p}_{\max}/(e \cdot n)$ and if we pick a price within $(1 - \varepsilon)$ of $p^*$ we get a $(1 - \varepsilon)$-approximation of $\widehat{R}_{1\text{-}\mathcal{PR}}$.

Next we show that for two rounds the problem becomes APX-hard. That is it is NP-hard to approximate better than within a factor of 259/260. The proof is via a reduction from the dominated set problem (see Lemma 9 in the appendix). In this reduction a client has valuation 1 iff at least one of its neighbors is in the dominating set, and 0 otherwise. This function is not additive and thus the result requires the DS-model (and not the SA-model).

**Theorem 7.** *Maximizing the revenue of* $2\text{-}\mathcal{PR}$*, resp.* $2\text{-}\mathcal{PR}^{IE}$*, is* APX*-hard for the DS-model (and also for the concave graph models of [9, 11]), in particular, it is not approximable within* 259/260*.*

Next we present the constant factor approximation of $\widehat{R}$ for $k \in \Omega(\log n)$. In the following strategy the set $A$ of clients for the first round is given. We will then choose $A$ using the $2\text{-}\mathcal{PD}^{\text{IE}}$-strategy of Theorem 2 to get the final result.

**Algorithm 4 (Strategy** $k$-PR$(c, A)$**).** Let $A \subseteq V$ be the influence set and $c > 1$ be a constant.

1. Give the product to all clients in $A$ for free in the first round.
2. Set $(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_t)$ to the myopic prices of the clients in $V \setminus A$ for the influence set $A$ in descending order.
3. Set $S_j = \left\{ i \mid \frac{\hat{p}_1}{c^{j-1}} \geq \hat{p}_i > \frac{\hat{p}_1}{c^j} \right\}$ and select the first $k - 1$ non-empty sets.
4. Each of these sets $S_j$ becomes a set of clients that is offered the product in one round with uniform price $p(j) = \min_{i \in S_j} \hat{p}_i$.

Our analysis of this strategy only collects revenue for clients in $V \setminus A$ and only exploits externalities induced by clients in $A$, i.e., from clients in the first round. Thus this algorithm would have the same performance if all nodes in $V \setminus A$ are offered the item in the same round but with $k - 1$ different prices.

We denote by $k$-PR$^*(c, q)$ the strategy where the influence set $A$ is chosen randomly such that each client is in $A$ with probability $q$, independently of the other clients. For the clients in the selected sets $S_j$ we extract at least $1/c$ of the revenue the optimal $2\text{-}\mathcal{PD}$ strategy would extract from them. Additional in each set there is one client that gets his myopic price (i.e., an additional $(1 - 1/c)$ factor of his optimal revenue). We show in the proof below that this second contribution can be used to compensate for the optimal revenue of the clients which are not in a selected set, resulting in the bound of the next theorem.

**Theorem 8.** *Let* $c > 1$ *be a constant. Given valuations from the SM-model with monotone hazard rates then for every* $2\text{-}\mathcal{PD}^{IE}$ *strategy* $s$ *with influence set* $A$ *the strategy* $(k+1)$-PR$(c, A)$ *achieves at least* $\min\{\frac{1}{c}, \frac{(c^k - 1)}{(c^k - 1) + e(n-k)(c-1)}\}$ *of the revenue* $R(s)$ *of* $s$*.*

*Proof.* Consider the set $V \setminus A = \{1, \ldots, n\}$ and let $\hat{\mathbf{p}} := (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n)$ be the vector of (myopic) prices induced by $A$. W.l.o.g., we assume that the prices and the corresponding clients are sorted in a descending order. By the definition $R(s) = \sum_{i=1}^{n} \hat{p}_i \cdot \mathbf{P}[v_i \geq \hat{p}_i]$.

For each client $i$ we denote the price charged by the strategy $(k+1)$-PR$(c, A)$ by $p_i^*$. The uniform price in round $j$ is denoted by $p(j)$. Then the revenue of $(k+1)$-PR$(c, A)$ is given by: $R((k+1)\text{-PR}(c, A)) = \sum_{i=1}^{n} p_i^* \cdot \mathbf{P}[v_i \geq p_i^*]$
Now, by construction of $S_j$, either $p_i^* \geq \hat{p}_i/c$ in the case where $i$ is in one of the selected sets (the first $k$ non-empty sets), or $p_i^* = 0$ otherwise.

Let $J$ be the set of the indices of the selected sets and $l$ the largest index among them. Let $m$ be the number of clients that are offered the product, i.e., $m$ is the client with the lowest myopic price in $S_l$.

Consider suitable chosen $\alpha \leq 1/c$. For each client $i$ in a selected set the algorithm collects at least a revenue of $\alpha \cdot \hat{p}_i \mathbf{P}[v_i \geq \hat{p}_i]$. Additionally for each $j \in J$ there exists at least one client $i_j \in S_j$ who is charged his myopic price and thus the algorithm collects the full revenue $\hat{p}_{i_j} \mathbf{P}\left[v_{i_j} \geq \hat{p}_{i_j}\right]$.

$$R((k+1)\text{-PR}(c, A)) \geq \sum_{1 \leq i \leq m} \alpha \hat{p}_i \cdot \mathbf{P}[v_i \geq \hat{p}_i] + (1-\alpha) \sum_{j \in J} p(j) \cdot \mathbf{P}\left[v_{i_j} \geq p(j)\right]$$

The first term is an $\alpha$-approximation for the revenue of the first $m$ clients. We next relate the second term to the revenue of the remaining clients and compute an approximation factor $\alpha$ such that:

$$(1-\alpha) \sum_{j \in J} p(j) \cdot \mathbf{P}\left[v_{i_j} \geq p(j)\right] \geq \alpha \sum_{m+1 \leq i \leq n} \hat{p}_i \cdot \mathbf{P}[v_i \geq \hat{p}_i]$$

By the definition of the sets, $p(j) \geq \frac{\hat{p}_1}{c^j}$ and by the monotone hazard rate condition $\mathbf{P}\left[v_{i_j} \geq p(j)\right] \geq 1/e$. Thus

$$\sum_{j \in J} p(j) \cdot \mathbf{P}\left[v_{i_j} \geq p(j)\right] \geq \sum_{l-k+1 \leq j \leq l} \hat{p}_1 c^{-j} \frac{1}{e} = c^{k-l} \cdot \frac{\hat{p}_1(1 - c^{-k})}{(c-1)e} = \frac{\hat{p}_1(c^k - 1)}{(c-1)c^l e}.$$

Using (a) $m \geq k$, (b) $\mathbf{P}[v_i \geq \hat{p}_i] \leq 1$ and (c) $\hat{p}_i \leq \hat{p}_1 \cdot c^{-l}$ for all $i \geq m+1$ we get $\sum_{i=m+1}^{n} \hat{p}_i \cdot \mathbf{P}[v_i \geq \hat{p}_i] \leq (n-k)\hat{p}_1 c^{-l}$. When resolving the inequality $\frac{(1-\alpha)\hat{p}_1(c^k-1)}{(c-1)c^l e} \geq \frac{\alpha(n-k)\hat{p}_1}{c^l}$ we obtain $\alpha \leq \frac{(c^k-1)}{(c^k-1)+e\cdot(n-k)(c-1)} =: \beta$. Now setting $\alpha = \min(\beta, 1/c)$ yields the claim. □

If the number of rounds is $\Omega(\log n)$ and we are using the influence set from the 2-$\mathcal{PD}^{\text{IE}}$ strategy 2-PD(0.5) in Theorem 2 we get a constant factor approximation.

**Corollary 2.** *Assuming valuations from the SM-model with monotone hazard rates,* $R(((\log_c n) + 1)\text{-PR}^*(c, 1/2)) \geq \widehat{R}/(4c \cdot e)$ *for any constant* $c > 1$.

*Proof.* Consider the 2-$\mathcal{PD}^{\text{IE}}$ strategy $s$ from Theorem 2, i.e., 2-PD(0.5), which is a 1/4-approximation of $\widehat{R}$, i.e., $R(s) \geq \widehat{R}/4$. Using the influence set $A$ from

$s$, i.e., randomly pocking nodes with probability 1/2, we get that $k$-PR$(c, A)$ is equal to $k$-PR$^*(c, 1/2)$. If we set $k = \log_c n + 1$ then Theorem 8 shows that $R(((\log_c n) + 1)\text{-PR}^*(c, 1/2)) \geq R(S)/(c \cdot e)$. Combining the two results we get a $\frac{1}{4e \cdot c}$-approximation of $\widehat{R}$. □

To obtain a $k$-$\mathcal{PR}$ strategy that matches the bound of Theorem 5 one can first construct the $((\log_c n) + 1)$-PR$^*(c, 1/2)$ strategy from above. Then for the $k$-$\mathcal{PR}$ strategy one uses the same influence set for the first round and for the remaining rounds one picks the $k - 1$ rounds with the highest expected payment in the strategy $((\log_c n) + 1)$-PR$^*(c, 1/2)$.[3]

Now let us consider 2-$\mathcal{PR}$ strategies. Due to the results in Section 4, the best we can hope for is a constant factor approximation of $\widehat{R}_{2\text{-}\mathcal{PR}}$, *not* of $\widehat{R}$ or of $\widehat{R}_{2\text{-}\mathcal{PD}}$. We first show that we can restrict ourselves to approximating the optimal IE strategy, as a revenue optimal IE strategy is within half of the revenue optimal 2-$\mathcal{PR}$ strategies. The proof idea is to design two IE strategies, one for the case that at least half of the revenue of the optimal $k$-$\mathcal{PR}$ strategy comes from the first round, and one or the case that it does not. In either case, at most half of the revenue is lost.

**Lemma 3.** *Given valuations $v_i$ from the SM-model, then $\widehat{R}^{IE}_{k\text{-}\mathcal{PR}} \geq \frac{1}{2}\widehat{R}_{k\text{-}\mathcal{PR}}$.*

Thus it suffices to approximate $\widehat{R}^{\text{IE}}_{k\text{-}\mathcal{PR}}$. We give a simple 2-round algorithm for the SM-model that is based on our 1-round strategy from Algorithm 3.

**Algorithm 5 (Strategy PR$^{0.5}(\varepsilon)$).** Let $\varepsilon$ be in $\mathbb{R}_{>0}$.

1. Assign each client in $V$ to an influence set $A$, s.t. each client is a member of $A$ independently of the others with probability 1/2. Give the product to the clients in $A$ for free in the first round.
2. Use Algorithm 3 to compute a $(1 - \varepsilon)$-approximation of the optimal revenue for the given influence set $A$.

In the analysis of Algorithm 5 we first bound the *probability* that the valuation of a client is larger than a fixed uniform price. This is different from the approach in Section 3, where it was sufficient to argue about the expected revenue we collect from a client. Then we use a technique similar to [11] to show that $R(\text{PR}^{0.5}(\varepsilon))$ is a $(1/8 - \varepsilon)$-approximation of $\widehat{R}^{\text{IE}}_{2\text{-}\mathcal{PR}}$.

**Theorem 9.** *Given valuations $v_i$ from the SM-model with monotone hazard rates, then $R(\text{PR}^{0.5}(\varepsilon)) \geq \left(\frac{1}{8} - \varepsilon\right) \cdot \widehat{R}^{IE}_{2\text{-}\mathcal{PR}}$ for every $\varepsilon > 0$.*

Together with Lemma 3 we then obtain that the above algorithm achieves at least 1/16 of $\widehat{R}_{2\text{-}\mathcal{PR}}$ in the SM-model with monotone hazard rates. By Theorem 5, $\widehat{R}^{\text{IE}}_{2\text{-}\mathcal{PR}}$ is a $\frac{1}{eH_n}$-approximation of $\widehat{R}^{\text{IE}}_{2\text{-}\mathcal{PD}}$ the above strategy is thus also a $\Theta(\frac{1}{H_n})$-approximation of $\widehat{R}$.

**Corollary 3.** *Given valuations $v_i$ from the SM-model with monotone hazard rates, then $R(\text{PR}^{0.5}(\varepsilon)) \geq \frac{(1/8-\varepsilon)}{4H_n} \cdot \widehat{R}^{IE}_{2\text{-}\mathcal{PD}} \geq \frac{(1/8-\varepsilon)}{16H_n} \cdot \widehat{R}$ for every $\varepsilon > 0$.*

[3] The authors are grateful to an anonymous reviewer for pointing this out.

The above results can be generalized to $k$-$\mathcal{PR}$ strategies.

**Theorem 10.** *Given valuations $v_i$ from the SM-model with monotone hazard rates, for any $k \geq 2$ and $\varepsilon > 0$ there exists a polynomial-time computable $k$-$\mathcal{PR}$ strategy $s$ such that $R(s) \geq \left(\frac{1}{8} - \varepsilon\right) \cdot \widehat{R}^{IE}_{k\text{-}\mathcal{PR}}$ and thus $R(s) \geq \left(\frac{1}{16} - \varepsilon\right) \cdot \widehat{R}_{k\text{-}\mathcal{PR}}$.*

The proof of Theorem 10 exploits that (i) we only lose a constant factor when ignoring the influence between the latter rounds and that (ii) when given the influence set and an instance without influence between the other clients we can give an approximation scheme for the optimal uniform prices and compute the corresponding round assignment.

Mirrokni et al. [11] study a different less general model, called concave graph model (CG). They give an algorithm that, under certain assumptions, finds an influence set $A$ which achieves at least $1/2$ of the revenue achieved by the optimal influence set. We extend this result to 2-$\mathcal{PR}^{\text{IE}}$ strategies in the SM-model. The key ideas of the proof are that (1) if the seller charges a uniform price sufficiently close to the optimal price then she only loses an $\varepsilon$ of the revenue and (2) once the seller has fixed the posted price, under the assumptions of the theorem, the expected revenue is a submodular function of the influence set.

**Theorem 11.** *Let $\mu > 0$, $M_v \geq 0$ and $M_p \geq 0$ such that $M_p \leq M_v \cdot \mu$. For each client, let their valuations follow the SM-model with the following additional assumptions: $g_i(\emptyset) \geq \mu > 0$, and $\tilde{v}_i$ is drawn from a probability distribution $F_i$ whose probability density function $f_i$ is positive, differentiable, non-decreasing on $(0, M_v)$ and for all $x \in (0, M_v)$, $f_i(x) \leq \bar{f}$ for some constant $\bar{f}$, and $F_i(0) = 0$, $F_i(M_v) = 1$. Let the price be in the interval $0 \leq p \leq M_p$. Then for every $\varepsilon = o(|V|^{-1})$ there is an algorithm finding a 2-$\mathcal{PR}^{IE}$ strategy $s$, i.e., an influence set $A^*$ and a uniform price $p^*$ for the second round, such that $R(s) \geq \left(\frac{1}{2} - \varepsilon\right) \cdot \widehat{R}^{IE}_{2\text{-}\mathcal{PR}}$.*

By Lemma 3, the algorithm of Theorem 11 achieves at least $(1/4 - \varepsilon)$ of $\widehat{R}_{2\text{-}\mathcal{PR}}$.

## 6 Extensions of the Model

Our models and results can be extended in several directions.

First, one can consider more general classes of externalities. In the *General Monotone Model (GM)* the valuation $v_i(B)$ of $i$ is drawn from a (known) distribution with the CDF $F_{i,B}$ such that $\mathbf{P}\left[v_i(B) \geq p\right] \geq \mathbf{P}\left[v_i(B') \geq p\right]$ for all $B' \subseteq B$ and $p \in \mathbb{R}_{\geq 0}$. Notice that the proofs of Theorems 5 and 8 do not exploit the fact that the valuations are from the SM-model and thus extend to the GM-model.

Second, the monotone hazard rate condition can be relaxed. For all theorems, except Theorem 3, the crucial part we use is that there is a myopic price with a certain acceptance probability. The exact approximation bound then depends on this acceptance probability. If one can guarantee a higher acceptance probability than $1/e$ also the presented approximation guarantees improve. For instance, if one considers only uniform distributions for $\tilde{v}_i$ then the acceptance probability of the myopic price is 1/2 and the approximation guarantee of Theorem 1 improves to $(k-1)/(3k-2)$.

Third, one can consider different classes of marketing strategies. For instance strategies where clients are split into $k$ groups and $l$ rounds such that (a) each client belongs to exactly one group and round, (b) all clients in the same group are offered the same price independent of their round and (c) only clients that have purchased the product *in a previous* round can influence the valuation functions of the clients in the current round, but this influence is independent of their group. For instance, a group could model all clients that live in the same country, preferred clients, or an age group. Our results extend to this setting as well. That is, one can get a constant factor of $\widehat{R}$ using $O(\log n)$ different groups/prices, but not with $o(\log n)$ different groups/prices.

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## A Proofs of Section 3

We will make use of the following two lemmata.

**Lemma 4 ([9]).** *Consider a monotone submodular function* $f : 2^X \to \mathbb{R}$ *and a subset* $S \subseteq X$*. Consider a random set* $S'$ *by choosing each element of* $S$ *independently with probability at least* $\beta$*. Then* $\mathbf{E}[f(S')] \geq \beta \cdot f(S)$*.*

**Lemma 5.** *Given the SM-model it holds that* $\hat{p}_i(S) = \hat{p}_i(V \setminus \{i\}) \cdot g_i(S)$ *for all* $S \subseteq V \setminus \{i\}$*.*

*Proof.* Note that $v_i(V \setminus \{i\}) = \tilde{v}_i \cdot g_i(V \setminus \{i\}) = \tilde{v}_i$ and that $\tilde{v}_i = v_i(S)/g_i(S)$. By the definition of the myopic price $\hat{p}_i(S)$, we know that setting $p = \hat{p}_i(S)$ maximizes the revenue $p \cdot \mathbf{P}[v_i(S) \geq p] = g_i(S)\left(\frac{p}{g_i(S)}\mathbf{P}\left[v_i(V \setminus \{i\}) \geq \frac{p}{g_i(S)}\right]\right)$. Since $g_i(S)$ does not depend on $p$, it follows that setting $p' = \frac{\hat{p}_i(S)}{g_i(S)}$ maximizes $p'\,\mathbf{P}[v_i(V \setminus \{i\}) \geq p']$. Thus, $\hat{p}_i(V \setminus \{i\}) = \frac{\hat{p}_i(S)}{g_i(S)}$ and $\hat{p}_i(S) = \hat{p}_i(V \setminus \{i\}) \cdot g_i(S)$. □

**Proof of Lemma 2.** For the random set $S$ it holds that

$$\mathbf{E}_S[\pi_i(S)] = \tag{1}$$
$$\mathbf{E}_S[\hat{p}_i(S) \cdot \mathbf{P}[v_i(S) \geq \hat{p}_i(S)]] = \tag{2}$$
$$\mathbf{E}_S[\hat{p}_i(V \setminus \{i\}) \cdot g_i(S)\mathbf{P}[v_i(V \setminus \{i\}) \geq \hat{p}_i(V \setminus \{i\})]] = \tag{3}$$
$$\mathbf{E}_S[g_i(S)] \cdot \hat{p}_i(V \setminus \{i\}) \cdot \mathbf{P}[v_i(V \setminus \{i\}) \geq \hat{p}_i(V \setminus \{i\})] \geq \tag{4}$$
$$\beta \cdot \pi_i(V \setminus \{i\}). \tag{5}$$

The equality of (3) follows from Lemma 5; the equality of (4) follows since only $g_i(S)$ depends on $S$ (and not on $p$). From Lemma 4 we get $\mathbf{E}_S[g_i(S)] \geq \beta \cdot g_i(V \setminus \{i\}) = \beta$, and thus, (5) follows. □

**Proof of Theorem 2.** We use $B_\ell$ to denote the clients that become active in round $\ell$ and $\ell(i)$ to denote the round in which client $i$ is offered the product. The probability that for two given clients $i \neq j \in V$ it holds that $j \in \bigcup_{\ell<\ell(i)} B_\ell$ under the condition that $i \notin V_1$ is $\mathbf{P}[j \in V_1] + \mathbf{P}[j \notin V_1] \cdot \mathbf{P}\left[j \in \bigcup_{1<\ell<\ell(i)} B_\ell \,\middle|\, i,j \notin V_1\right]$. It holds that $\mathbf{P}[j \in V_1] = q$ and $\mathbf{P}[j \notin V_1] = 1-q$. $\mathbf{P}\left[j \in \bigcup_{1<\ell<\ell(i)} B_\ell \,\middle|\, i,j \notin V_1\right]$ is given by the probability that $\ell(j) < \ell(i)$ times by the probability that client $j$ accepts the offer. Furthermore $\mathbf{P}[\ell(j) < \ell(i) \mid i,j \notin V_1] = \frac{k-2}{k-1}\frac{1}{2}$ because $\ell(i) < \ell(j)$ and $\ell(j) < \ell(i)$ have equal probability and with probability $\frac{k-2}{k-1}$ it holds that $i$ and $j$ are not assigned to the same round. Finally, note that we offer the product for the myopic price and thus, by the monotone hazard rate condition, the acceptance probability of $j$ is at least $1/e$. Thus, $\beta := q + (1-q)\frac{k-2}{k-1}\frac{1}{2}\frac{1}{e}$ is a lower bound of the probability that $j \in \bigcup_{\ell<\ell(i)} B_\ell$ under the condition that $i \notin V_1$.

Let $S_i$ be $\bigcup_{\ell<\ell(i)} B_\ell$ for all $i \in V$. It follows from Lemma 2, the definition of $\beta$, and $\sum_{i \in V} \pi_i(V \setminus \{i\}) \geq \widehat{R}$ that $R(k\text{-PD}(q)) = \sum_{i \in V} \mathbf{P}[i \notin V_1] \cdot \mathbf{E}[\pi_i(S_i) \mid i \notin V_1] \geq \sum_{i \in V} \mathbf{P}[i \notin V_1] \cdot \beta \cdot \pi_i(V \setminus \{i\}) = \sum_{i \in V}(1-q) \cdot \beta \cdot \pi_i(V \setminus \{i\}) \geq (1-q) \cdot \beta \cdot \widehat{R}$; the maximal value of $(1-q) \cdot \beta$ is $\frac{e \cdot (k-1)}{4e(k-1)-2k+4}$ and can be obtained by setting $q = 1 - \frac{e \cdot (k-1)}{2e(k-1)-k+2}$. □

### A.1 Proof of Theorem 3

For a proof of Theorem 3 we need the following lemmas.

**Lemma 6 ([3]).** *Given a distribution function $F$ satisfying (i) the monotone hazard rate condition and (ii) $F(0-) = 0$ then $\frac{1}{x}\log(1 - F(x))$ and $[1 - F(x)]^{\frac{1}{x}}$ are non-increasing in $x$.*

The following lemma generalizes Lemma 1 for discounted prices.

**Lemma 7.** *Assume the SM-model with the monotone hazard rates, then $\mathbf{P}[v_i(B) \geq \alpha \cdot \hat{p}_i(B)] \geq \mathbf{P}[v_i(B) \geq \hat{p}_i(B)]^\alpha \geq \left(\frac{1}{e}\right)^\alpha$.*

*Proof.* Consider the distribution $F$ of valuation $v_i$. Then $\mathbf{P}[v_i(B) \geq \alpha \cdot \hat{p}_i(B)] = 1 - F(\alpha \cdot \hat{p}_i(B))$. By Lemma 6, $(1 - F(\alpha \cdot \hat{p}_i(B)))^{\frac{1}{\alpha}} \geq (1 - F(p_i(B))$. Now as $\mathbf{P}[v_i(B) \geq \hat{p}_i(B)] = (1 - F(p_i(B))$ we obtain the first inequality. For the second inequality recall that by the monotone hazard rate condition and Lemma 1, $\mathbf{P}[v_i(B) \geq \hat{p}_i(B)] \geq 1/e$. □

To analyze the strategy we again consider the expected payment $\pi_{i,\alpha}(S) = \alpha \cdot \hat{p}_i(S) \cdot \mathbf{P}[v_i(S) \geq \alpha \cdot \hat{p}_i(S)]$ the seller gets from client $i$ when giving a discount factor $\alpha$. we can extract from a client $i$ given the active clients $B$. Next we generalize Lemma 2 for $k$-PD$(\bar{q}, \bar{\alpha})$ strategies.

**Lemma 8.** *For $S \subseteq V \setminus \{i\}$: $\pi_{i,\alpha}(S) \geq \alpha \cdot e^{1-\alpha}\pi_i(S)$.*

*Proof.* Consider the expected payment $\pi_{i,\alpha}(S)$ from client $i$ when getting a discount factor of $\alpha$. Using Lemma 7 we get $\pi_{i,\alpha}(S) \geq \alpha \cdot \hat{p}_i(S) \cdot \mathbf{P}[v_i(S) \geq \hat{p}_i(S)]^\alpha$ and by the definition of $\pi_i(S)$ the latter is equal to $\alpha \cdot \mathbf{P}[v_i(S) \geq \hat{p}_i(S)]^{\alpha-1} \pi_i(S)$. Again applying Lemma 7 results $\pi_{i,\alpha}(S) \geq \alpha \cdot (1/e)^{\alpha-1}\pi_i(S)$. □

**Proof of Theorem 3.** Let $k$ be the size of the vectors, then $R(\text{PD}(\bar{q}, \bar{\alpha})) = \sum_{i \in V} \sum_{j=1}^{k} \mathbf{P}[i \in V_j] \cdot \mathbf{E}_{S_l}[\pi_{i,\alpha}(S_j)]$. The probability $\beta_j$ that an arbitrary client $i$ is active in round $j$ is given by the probability that $i$ is in one of the previous rounds and buys the product. Thus $\beta_j \geq \sum_{l=1}^{j-1} q_l \cdot \left(\frac{1}{e}\right)^{\alpha_l}$. Next we use the presented lemmas and linearity of expectation to relate the revenue of PD$(\bar{q}, \bar{\alpha})$

to $\widehat{R}$.

$$
\begin{aligned}
R(\mathrm{PD}(\bar{q},\bar{\alpha})) &\geq \sum_{i\in V}\sum_{j=1}^{k} q_j \cdot \mathbf{E}_{S_l}\left[\alpha_j \cdot e^{1-\alpha_j} \cdot \pi_i(S_j)\right] \\
&\geq \sum_{i\in V}\sum_{j=1}^{k} q_j \cdot \left(\sum_{l=1}^{j-1} q_l \cdot \left(\frac{1}{e}\right)^{\alpha_l}\right) \cdot \alpha_j \cdot e^{1-\alpha_j} \cdot \pi_i(V\setminus\{i\}) \\
&\geq \sum_{j=1}^{k} q_j \cdot \left(\sum_{l=1}^{j-1} q_l \cdot \left(\frac{1}{e}\right)^{\alpha_l}\right) \cdot \alpha_j \cdot e^{1-\alpha_j} \cdot \sum_{i\in V}\pi_i(V\setminus\{i\}) \\
&\geq \sum_{j=1}^{k} q_j \cdot \left(\sum_{l=1}^{j-1} q_l \cdot \left(\frac{1}{e}\right)^{\alpha_l}\right) \cdot \alpha_j \cdot e^{1-\alpha_j} \cdot \widehat{R}
\end{aligned}
$$

Hence to maximize the approximation guarantee we consider

$$
\sum_{j=1}^{k} q_j \cdot \left(\sum_{l=1}^{j-1} q_l \cdot \left(\frac{1}{e}\right)^{\alpha_l}\right) \cdot \alpha_j \cdot e^{1-\alpha_j-1}.
$$

Certain assignments to $\bar{q}$, $\bar{\alpha}$ and the corresponding approximation ratios can be found in Table 1. □

**Table 1.** Optimal values $q, \alpha$ for $k$-PD$(\bar{q},\bar{\alpha})$ strategies and the corresponding approximation ratio.

| $k$ | $\bar{q}$ | $\bar{\alpha}$ | approx |
|---|---|---|---|
| 2 | (0.5,0.5) | (0,1) | 0.25 |
| 3 | (0.432,0.267,0.299) | (0,0.757,1) | 0.279 |
| 4 | (0.399,0.187,0.199,0.213) | (0,0.644,0.850,1) | 0.291 |
| 5 | (0.369,0.138,0.151,0.164,0.176) | (0,0.5,0.707,0.866,1) | 0.298 |
| 6 | (0.353,0.113,0.120,0.129,0.137,0.145) | (0,0.447,0.632,0.774,0.894,1) | 0.302 |
| 7 | (0.342,0.096,0.100,0.106, 0.112,0.118,0.124) | (0,0.408,0.577,0.707, 0.816,0.912,1) | 0.306 |
| 8 | (0.333,0.084,0.086,0.090, 0.0946,0.0992,0.103,0.108) | (0,0.377,0.534,0.654, 0.755,0.845,0.925,1) | 0.308 |
| 9 | (0.326,0.075,0.075,0.0783,0.081, 0.085,0.088,0.092,0.095) | (0,0.353,0.5,0.612,0.707, 0.790,0.866,0.935,1) | 0.310 |
| 10 | (0.320,0.068,0.067,0.069,0.071, 0.074, 0.077, 0.080,0.083,0.085) | (0,0.333,0.471,0.577,0.666, 0.745,0.816,0.881,0.942,1) | 0.311 |

## B Proofs of Section 4

**Proof of Theorem 4.** (1) Consider the following example of the marketing problem. The clients are given by $V = \{1,\ldots,n\}$, and for given parameter $\lambda$

with $0 < \lambda < 1$, and for each $B \subset V$ the valuation functions $v_i$ are given by

$$\mathbf{P}\left[v_i(B) \geq p\right] = \begin{cases} 1 & \text{for } p \leq 0, \\ \lambda^{i-1} & \text{for } 0 < p \leq \hat{p}_i, \\ 0 & \text{for } p > \hat{p}_i, \end{cases}$$

with $\hat{p}_i := (\sum_{j=i}^{n} \lambda^{j-1})^{-1}$. Thus, the valuations of the clients are not influenced by the other clients, and $\widehat{R}_{k\text{-}\mathcal{PD}}$ does not depend on $k$ ($\widehat{R}_{k\text{-}\mathcal{PD}} = \widehat{R}_{1\text{-}\mathcal{PD}}$). Note that for each client $i$, $\hat{p}_i$ is the optimal myopic price, since for any price $p$ with $0 < p < \hat{p}_i$ it holds that $\mathbf{P}\left[v_i \geq p\right] = \mathbf{P}\left[v_i \geq \hat{p}_i\right]$, and hence $p \cdot \mathbf{P}\left[v_i \geq p\right] \leq \hat{p}_i \cdot \mathbf{P}\left[v_i \geq \hat{p}_i\right]$. The revenue with price discrimination and parameter $\lambda$ is then

$$R_{\mathcal{PD}}^{\lambda} := \sum_{i=1}^{n} \hat{p}_i \cdot \mathbf{P}\left[v_i \geq \hat{p}_i\right] = \sum_{i=1}^{n} \frac{\lambda^{i-1}}{\sum_{j=i}^{n} \lambda^{j-1}}.$$

As there is no influence between clients we have that $\widehat{R}_{k\text{-}\mathcal{PR}} \leq k \cdot \widehat{R}_{1\text{-}\mathcal{PR}}$. Hence, we first consider $\widehat{R}_{1\text{-}\mathcal{PR}}$. The optimal uniform prices have to be in the set $\{\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n\}$ for the same reason as these are optimal myopic prices: using a price $p$ such that $\hat{p}_{i-1} < p < \hat{p}_i$ does not increase the probability $\mathbf{P}\left[v_j \geq p\right]$ for any client $j$, but it does decrease the revenue since the price is lower.

The revenue for any uniform price $\hat{p}_i \in \{\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n\}$ is

$$R_{1\text{-}\mathcal{PR}}^{\hat{p}_i} := \hat{p}_i \cdot \sum_{j=1}^{n} \mathbf{P}\left[v_j \geq \hat{p}_i\right] = \frac{1}{\sum_{j=i}^{n} \lambda^{j-1}} \cdot \sum_{j=i}^{n} \lambda^{j-1} = 1.$$

Thus, all the prices in the set $\{\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n\}$ are optimal uniform prices, and $\widehat{R}_{1\text{-}\mathcal{PR}} = R_{1\text{-}\mathcal{PR}}^{\hat{p}_i}$. The limit of the ratio $\frac{\widehat{R}_{1\text{-}\mathcal{PR}}}{R_{\mathcal{PD}}^{\lambda}}$ for $\lambda \to 0_+$ is

$$\lim_{\lambda \to 0_+} \frac{\widehat{R}_{1\text{-}\mathcal{PR}}}{R_{\mathcal{PD}}^{\lambda}} = \frac{1}{\lim_{\lambda \to 0_+} \sum_{i=1}^{n} \frac{\lambda^{i-1}}{\sum_{j=i}^{n} \lambda^{j-1}}} = \frac{1}{n}.$$

Hence, for any $\varepsilon > 0$ we can find a $\lambda \in (0,1)$ and a value $n_0$ such that for $n > n_0$ it holds that

$$\frac{1}{k} \widehat{R}_{k\text{-}\mathcal{PR}} \leq \widehat{R}_{1\text{-}\mathcal{PR}} \leq \frac{1+\varepsilon}{n} R_{\mathcal{PD}}^{\lambda} = \frac{1+\varepsilon}{n} \widehat{R}_{k\text{-}\mathcal{PD}}.$$

(2) For each $i$ let $\hat{p}_i$ denote the myopic price $\hat{p}_i(\emptyset)$. The revenue for the optimal 1-$\mathcal{PD}$ strategy is given by $\widehat{R}_{1\text{-}\mathcal{PD}} = \sum_{i \in V} \hat{p}_i \cdot \mathbf{P}\left[v_i \geq \hat{p}_i\right]$. As uniform price $p^*$ we set $p^* = \hat{p}_j$ such that $j \in \operatorname*{argmax}_{i \in V} \hat{p}_i \cdot \mathbf{P}\left[v_i \geq \hat{p}_i\right]$. Hence the revenue $\widehat{R}_{1\text{-}\mathcal{PR}}$ is given by $\widehat{R}_{1\text{-}\mathcal{PR}} = \sum_{i \in V} p^* \cdot \mathbf{P}\left[v_i \geq p^*\right] \geq \max_{i \in V} \hat{p}_i \cdot \mathbf{P}\left[v_i \geq \hat{p}_i\right] \geq \frac{1}{|V|} \cdot \widehat{R}_{1\text{-}\mathcal{PD}}$.

(3) For $\widehat{R}_{2\text{-}\mathcal{PR}}^{\text{IE}} \geq \frac{1}{n} \widehat{R}_{2\text{-}\mathcal{PD}}^{\text{IE}}$, consider the optimal 2-$\mathcal{PD}^{\text{IE}}$ strategy with influence set $A$ and a 2-$\mathcal{PR}^{\text{IE}}$ strategy with the same influence set. By definition there is no revenue in the first round and in the second round we are faced with

the problem of approximating 1-$\mathcal{PD}$ with 1-$\mathcal{PR}$, according to (2) we can get a $1/n$ approximation. Finally for $\frac{1}{n}\widehat{R}^{\text{IE}}_{2\text{-}\mathcal{PD}} \geq \frac{1}{4n} \cdot \widehat{R}$ recall that, by Corollary 1, we have $\widehat{R}^{\text{IE}}_{2\text{-}\mathcal{PD}} \geq \frac{1}{4}\widehat{R}$. □

**Proof of Theorem 5.** (1) Consider the following example of the marketing problem with valuations from the SA-model. The clients are given by $V = \{1, \ldots, n\}$ and, for each $B \subset V$ the valuation functions $v_i$ are given by $v_i(B) = 1/i$. The total revenue with price discrimination is then $\widehat{R}_{1\text{-}\mathcal{PD}} = \sum_{i=1}^{n} \frac{1}{i} = H_n$.

The optimal uniform price is going to be of the form $1/i$: using a price $p$ such that $\frac{1}{i} < p < \frac{1}{i-1}$ does not increase the number of buyers, but it does decrease the revenue since the price is lower. The revenue $R_{1\text{-}\mathcal{PR}}(1/j)$ for a uniform price $1/j$ is then $(1/j) \cdot |\{i \in V \mid 1/i \geq 1/j\}| = 1$, and thus, $\widehat{R}_{1\text{-}\mathcal{PR}} = 1$. Moreover, as there is no influence between the clients we have that

$$\widehat{R}_{k\text{-}\mathcal{PR}} \leq k \cdot \widehat{R}_{1\text{-}\mathcal{PR}} \leq \frac{k}{H_n}\widehat{R}_{1\text{-}\mathcal{PD}} = \frac{k}{H_n}\widehat{R}_{k\text{-}\mathcal{PD}}.$$

(2) Let $\hat{\mathbf{p}} := (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n)$ be the vector of optimal myopic prices for the clients. W.l.o.g., we assume that the prices are sorted in a descending order. Consider the 1-$\mathcal{PR}$ strategy $s$ that selects the myopic price $\hat{p}_i$ with probability $\frac{1}{i \cdot H_n}$. Note that $\sum_{i=1}^{n} \frac{1}{i \cdot H_n} = \frac{1}{H_n} \cdot \sum_{i=1}^{n} \frac{1}{i} = 1$.

The expected revenue from using these uniform prices is then

$$\begin{aligned} R(s) &= \sum_{i=1}^{n} \frac{1}{i \cdot H_n} \cdot \sum_{j=1}^{n} \mathbf{P}[v_j \geq \hat{p}_i] \cdot \hat{p}_i \\ &\geq \sum_{i=1}^{n} \frac{1}{i \cdot H_n} \cdot \sum_{j=1}^{i} \mathbf{P}[v_j \geq \hat{p}_i] \cdot \hat{p}_i \\ &\geq \sum_{i=1}^{n} \frac{1}{i \cdot H_n} \cdot \sum_{j=1}^{i} \frac{1}{e} \cdot \hat{p}_i = \frac{1}{e \cdot H_n} \cdot \sum_{i=1}^{n} \hat{p}_i. \end{aligned}$$

At the same time, the revenue with price discrimination $\widehat{R}_{1\text{-}\mathcal{PD}}$ is $\sum_{i=1}^{n} \hat{p}_i \cdot \mathbf{P}[v_i \geq \hat{p}_i] \leq \sum_{i=1}^{n} \hat{p}_i$. Therefore, we get $\widehat{R}_{1\text{-}\mathcal{PR}} \geq R(s) \geq \frac{1}{e \cdot H_n}\widehat{R}_{1\text{-}\mathcal{PD}}$.

(3) Consider the optimal 2-$\mathcal{PD}^{\text{IE}}$ strategy with influence set $A$ and a 2-$\mathcal{PR}^{\text{IE}}$ strategy with the same influence set. By definition there is no revenue in the first round and in the second round we are faced with the problem of approximating 1-$\mathcal{PD}$ with 1-$\mathcal{PR}$, according to (2) we can get a $\frac{1}{e \cdot H_n}$ approximation. □

## C Proofs of Section 5

**Proof of Theorem 6.** Consider the optimal uniform price $p$. Given the monotone hazard rate condition it is easy to show that $p \leq \hat{p}_{\max}$. We next show that also $p \geq \frac{\hat{p}_{\max}}{e \cdot n}$. Towards a contradiction assume that not. Then the revenue is bounded by $p \cdot n < \frac{\hat{p}_{\max}}{e}$. But using $\hat{p}_{\max}$ as unique price gives at least a revenue $\frac{\hat{p}_{\max}}{e}$, a contradiction to the optimality of $p$.

Given the above bounds we know that Algorithm 3 considers a price $p_j$ such that $p \geq p_j \geq p \cdot (1-\varepsilon)$. Let $R(p')$ denote the revenue achieved by the uniform price $p'$ then:

$$R(p_j) = \sum_{i \in V} p_j \cdot \mathbf{P}\left[v_i(\emptyset) \geq p_j\right] \geq \sum_{i \in V} (1-\varepsilon) p \cdot \mathbf{P}\left[v_i(\emptyset) \geq p\right] = (1-\varepsilon) \cdot R(p)$$

Hence, Algorithm 3 gives a $(1-\varepsilon)$ of $\widehat{R}_{1\text{-}\mathcal{PR}}$. □

**Lemma 9 (Maximum dominated set).** *The Maximum dominated set problem, i.e., computing a maximum set $S \subseteq V$ s.t. for each $b \in S$ there is an $a \in V \setminus S$ with $(a,b) \in E$, is* APX*-hard (not approximable within* $259/260$*).*

*Proof.* It is known that the Minimum dominating set problem for graphs with bounded degree is APX-hard [5]. Let us assume that for each $\varepsilon > 0$ there is an $(1-\varepsilon)$-approximation algorithm for the Maximum dominated set problem. Consider an arbitrary graph of degree at most $\Delta$ with $Opt_M$ being a minimum dominating set and $Opt_S = V \setminus Opt_M$ being the corresponding maximum dominated set. Further let $A_S$ be the dominated set returned by the approximation algorithm and $A_M = V \setminus A_S$ then $|A_M| = n - |A_S| \leq n - (1-\varepsilon) \cdot |Opt_S| = n - |Opt_S| + \varepsilon \cdot |Opt_S| = |Opt_M| + \varepsilon \cdot |Opt_S|$. Now as the graph has degree $\leq \Delta$ clearly $|Opt_S| \leq \Delta |Opt_M|$ and thus $|A_M| \leq (1 + \Delta\varepsilon) \cdot |Opt_M|$. With $\varepsilon = \delta/\Delta$, we would obtain $(1+\delta)$-approximation algorithms for Minimum Dominating Set, contradicting the APX-hardness. Using a result from [5], showing that Minimum dominating set on graphs with degree $\leq 5$ can not be approximated within $53/52$, we obtain the hardness of the Maximum dominated set problem for constants $< 259/260$. □

**Proof of Theorem 7.** This is by a reduction from the Maximum dominated set problem. For an instance $G = (V, E)$ of Maximum dominated set, consider a social network with the DS-model such that for all clients the valuations $v_i(B)$ are 1 if $B \cap N_i \neq \emptyset$ and 0 otherwise, where $N_i$ denotes the neighbors of $i$ in $G$. Now as we can extract at most revenue 1 from each client the optimal uniform price in this setting is always $p^* = 1$. Thus maximizing the total revenue is equivalent to computing a maximum dominated set which is APX-hard. Finally observe that the valuation functions $v_i$ can also be modeled in the convex graph models by a convex function $f(x) = x$ for $x < 1$ and $f(x) = 1$ for $x \geq 1$ and deterministic edge weights $w_{i,j} = 1$ for $(i,j) \in E$. □

**Proof of Lemma 3 .** Consider a $k$-$\mathcal{PR}$ strategy $s$ maximizing the sellers revenue, and let $V_1$ be the set of clients visited in the first round. Let $R_1$ be the expected revenue from the clients in $V_1$ and $R_2$ the expected revenue of the clients in $V \setminus V_1$. If $R_1 \leq R_2$ then consider the $k$-round IE strategy $s_1$ which coincides with $s$ except that the uniform price in the first round is zero. Clearly

$s_1$ has at least revenue $R_2$ and thus $R(s_1) \geq 1/2\ R(s)$. Otherwise if $R_1 > R_2$ then consider the IE strategy $s_2$ with influence set $V \setminus V_1$ and as second round the set $V_1$ with the prices from $s$. By the monotonicity of the valuation functions $s_2$ has at least revenue $R_1$ and thus $R(s_2) \geq 1/2\ R(s)$. □

**Proof of Theorem 9.** Assume that $\widehat{A}$ is an optimal influence set, $\widehat{B} = V \setminus \widehat{A}$ and $\hat{p}^*$ is an optimal uniform price. Let $A$ be the influence set chosen by $\mathrm{PR}^{0.5}(\varepsilon)$ and $B' := \widehat{B} \setminus A$.

We will show the following *Claim*, for all $i \in \widehat{B}$.

$$\mathbf{P}\left[v_i(A) \geq \frac{p}{2}\right] \geq \frac{1}{2} \cdot \mathbf{P}\left[v_i(V \setminus \{i\}) \geq p\right] \tag{6}$$

Recall that $v_i(A) = \tilde{v}_i \cdot g_i(A)$ for each $i \in V$. We know that since $g_i(A)$ is submodular, for each $A' \subseteq V$, $g_i(A') + g_i(V \setminus (A' \cup \{i\})) \geq g_i(V \setminus \{i\})$. Thus $\forall i \in V$, $\max\{g_i(A'), g_i(V \setminus (A' \cup \{i\}))\} \geq \frac{1}{2} g_i(V \setminus \{i\})$. It follows that with probability $\frac{1}{2}$, $g_i(A) \geq \frac{1}{2} g_i(V \setminus \{i\})$. Thus with probability $\frac{1}{2}$ a set $A$ was chosen such that $\frac{p}{g(V \setminus \{i\})} \geq \frac{p/2}{g_i(A)}$ and thus:

$$\mathbf{P}\left[v_i(A) \geq \frac{p}{2}\right] = \mathbf{P}\left[\tilde{v}_i \geq \frac{p/2}{g_i(A)}\right] \geq \frac{1}{2}\mathbf{P}\left[\tilde{v}_i \geq \frac{p}{g_i(V \setminus \{i\})}\right] = \frac{\mathbf{P}\left[v_i(V \setminus \{i\}) \geq p\right]}{2}$$

This proves the claim.

We can get positive revenue for $i \in B' = \widehat{B} \setminus A$. Now consider the $k$-$\mathcal{PR}$ strategy $s$ with influence set $A$ and uniform price $\frac{\hat{p}^*}{2}$. From the above we get:

$$R(s) \geq \mathbf{E}_A\left[\sum_{i \in B'} \frac{\hat{p}^*}{2} \mathbf{P}\left[v_i(A) \geq \frac{\hat{p}^*}{2}\right]\right] \geq \frac{\hat{p}^*}{2 \cdot 2} \sum_{i \in \widehat{B}} \frac{1}{2} \mathbf{P}\left[v_i(\widehat{A}) \geq \hat{p}^*\right] = \frac{\widehat{R}^{\mathrm{IE}}_{2\text{-}\mathcal{PR}}}{8}$$

This bound clearly also holds for the optimal uniform price. Theorem 6 shows that step 2 computes a uniform price that is within $(1-\varepsilon)$ of the optimal strategy. Thus the bound of the Theorem follows. □

### C.1 Proof of Theorem 10

We first give an algorithm that solves the sub-problem of, given an influence set for the first round, assigning the remaining clients to the remaining rounds and setting the corresponding uniform prices. This algorithm is presented as Algorithm 6 and analyzed in Lemma 10. In particular Algorithm 6 provides an approximation scheme for this sub-problem.

**Algorithm 6.** Let $c > 1$ and $\beta > 1$. Let $A \subseteq V$ be a set of clients having the product.

1. Compute $\hat{p}_{\max} = \max_{i \in V} \hat{p}_i(A)$.
2. Compute $P = \{\frac{\hat{p}_{\max}}{c^j} \mid 0 \leq j \leq \lceil \log_c(\beta \cdot e \cdot n) \rceil\}$
3. For each $S \subseteq P, |S| = k$
   (a) For $p \in S$ build a round $B$ and set $p$ as uniform price.
   (b) For $i \in V \setminus A$ assign $i$ to the round with the uniform price that maximizes the expected payment we get from $i$, assuming only influence from $A$.
   (c) Compute the expected revenue of this assignment.
4. Pick the set $S$ with the maximal expected revenue.

The algorithm is based on the following ideas: (i) If we know the uniform prices of the rounds then we can easily assign clients optimally to the rounds. (ii) If we enumerate all possible price vectors of size $k$ we will eventually consider the optimal strategy. (iii) We can ignore clients if their myopic prices are much smaller than the largest myopic price. (iv) We do not have to consider all possible uniform prices. If we ensure that for each possible price we have one price that is within a constant factor we only lose a constant factor of the revenue.

Exploiting these ideas we next show that Algorithm 6 provides an approximation scheme.

**Lemma 10.** *Consider the SM-model with monotone hazard rates and clients that do not have network externalities, i.e.,* $g_i(X) = 1$ *for all* $i$ *and* $X \subset V$. *Then for each* $\varepsilon > 0$ *we can set the parameters* $c$ *and* $\beta$ *such that Algorithm 6 with* $A = \emptyset$ *computes a* $(1 - \varepsilon)$*-approximation of the optimal revenue.*

*Proof.* Consider an optimal strategy $s$ with round assignment $(\widehat{B}_1, \ldots, \widehat{B}_k)$ and the corresponding uniform prices $(\hat{p}_1^*, \ldots, \hat{p}_k^*)$.

We first consider the loss we get by restricting our strategy to prices in the set $P = \{\frac{\hat{p}_{\max}}{c^j} \mid 0 \leq j \leq \lceil \log_c(\beta \cdot e \cdot n) \rceil\}$. We can transform $s$ to a new strategy as follows: First we can ignore all rounds with uniform price $\leq \frac{\hat{p}_{\max}}{\beta \cdot e \cdot n}$. As the optimal strategy has at least revenue $\frac{\hat{p}_{\max}}{e}$ this gives a factor of $(1 - \frac{1}{\beta})$. Second we can replace all the other uniform price $\hat{p}_l^*$ by the maximal $p \in P$ such that $p \leq \hat{p}_l^*$. By or choice of $P$ the new uniform prices are within a factor of $1/c$ of the original prices. Moreover as the price only goes down the acceptance probability is at least the same as in the original strategy and thus the revenue is only decreased by a factor of at most $1/c$. Thus, restricting the prices results a loss of at most $1/c \cdot (1 - \frac{1}{\beta})$.

Now notice that the size of the set of feasible prices $P$ only increases logarithmically with the input size. Hence the enumeration of all subsets of size $k$ is in polynomial time. Moreover as we do a full enumeration of all strategies with prices in $P$ we will find the optimal such strategy and thus get a $1/c \cdot (1 - \frac{1}{\beta})$ approximation, which can be made arbitrarily close to 1 by setting $c$ and $\beta$ accordingly. □

Next we present the strategy $\mathrm{PR}_k^{0.5}(\varepsilon)$ which finally will play the role of strategy $s$ in Theorem 10.

**Algorithm 7 (Strategy $\mathrm{PR}_k^{0.5}(\varepsilon)$).** Let $\varepsilon$ be in $\mathbb{R}_{>0}$.

1. Assign each client in $V$ to the influence set $A$, s.t. each client is a member of $A$ independently of the others with probability 1/2. Give the product to the clients in $A$ for free in the first round.
2. Use Algorithm 6 to compute a $(1-\varepsilon)$-approximation of the optimal revenue for the remaining $(k-1)$ rounds (given influence set $A$).

The idea of the below analysis of strategy $\mathrm{PR}_k^{0.5}(\varepsilon)$ is to only consider influence from the first round and ignoring influence between the remaining rounds. This allows to apply Lemma 10 to rounds $2,\dots,k$, after fixing the influence set.

**Proof of Theorem 10.** Consider strategy $\mathrm{PR}_k^{0.5}(\varepsilon)$. We show that $\mathrm{PR}_k^{0.5}(\varepsilon)$ is a $(1/8-\varepsilon)$-approximation of the optimal $k$-$\mathcal{PR}^{\mathrm{IE}}$ strategy.

Assume that $\widehat{A}$ is an optimal influence set, $(\widehat{B}_2,\dots,\widehat{B}_k)$ the optimal round assignment $(\bigcup_{l=2}^{k}\widehat{B}_l = V\setminus\widehat{A})$, and $(\hat{p}_2^*,\dots\hat{p}_k^*)$ the corresponding optimal uniform prices. Let $A$ be the influence set chosen by $\mathrm{PR}_k^{0.5}(\varepsilon)$ and $B_i' := \widehat{B}_i \setminus A$.

We will show the following *claim*, for all $2 \le l \le k$, for all $i \in \widehat{B}_l$, and for all $p \ge 0$.

$$\mathbf{P}\left[v_i(A) \ge \frac{p}{2}\right] \ge \frac{1}{2}\cdot \mathbf{P}\left[v_i(V\setminus\{i\}) \ge p\right] \tag{7}$$

Recall that $v_i(A) = \tilde{v}_i \cdot g_i(A)$ for each $i \in V$. We know that since $g_i(\cdot)$ is submodular, for each $A' \subseteq V$, $g_i(A') + g_i(V\setminus(A'\cup\{i\})) \ge g_i(V\setminus\{i\})$. Thus $\forall i \in V$, $\max\{g_i(A'), g_i(V\setminus(A'\cup\{i\}))\} \ge \frac{1}{2}g_i(V\setminus\{i\})$. It follows that with probability $\frac{1}{2}$, $g_i(A) \ge \frac{1}{2}g_i(V\setminus\{i\})$. Thus with probability $\frac{1}{2}$ a set $A$ was chosen such that $\frac{p}{g(V\setminus\{i\})} \ge \frac{p/2}{g_i(A)}$ and thus:

$$\mathbf{P}\left[v_i(A) \ge \frac{p}{2}\right] = \mathbf{P}\left[\tilde{v}_i \ge \frac{p/2}{g_i(A)}\right] \ge \frac{1}{2}\mathbf{P}\left[\tilde{v}_i \ge \frac{p}{g_i(V\setminus\{i\})}\right] = \frac{\mathbf{P}\left[v_i(V\setminus\{i\}) \ge p\right]}{2}$$

This proves the claim.

We can get positive revenue only for $i \notin A$. Now consider the $k$-$\mathcal{PR}$ strategy $s'$ with influence set $A$, rounds $(B_2',\dots,B_k')$, and uniform prices $\frac{\hat{p}_l^*}{2}$. From the above we get:

$$R(s') = \mathbf{E}_A\left[\sum_{l=2}^{k}\sum_{i\in B_l'}\frac{\hat{p}_l^*}{2}\mathbf{P}\left[v_i(A) \ge \frac{\hat{p}_l^*}{2}\right]\right] \ge \sum_{l=2}^{k}\frac{\hat{p}_l^*}{2\cdot 2}\sum_{i\in\widehat{B}_l}\frac{1}{2}\mathbf{P}\left[v_i(V\setminus\{i\}) \ge \hat{p}_l^*\right] \ge \frac{\widehat{R}_{k\text{-}\mathcal{PR}}^{\mathrm{IE}}}{8}$$

Notice that in the above analysis we did not use any influence between the latter rounds. Hence, while we cannot compute the strategy $s'$ itself, by Lemma 10, Algorithm 6 gives us a $(1-\varepsilon)$ approximation of it. Thus the bound of the theorem follows. □

### C.2 Proof of Theorem 11

Mirrokni et al. [11] show an algorithm which is able to find an influence set $A$ which achieves at least $1/2$ of the revenue achieved by the optimal influence set $\hat{A}$. The algorithm assumes that the prices are drawn from an interval $p \in [0, M_p]$. The expected revenue function, given the price $p$ and influence set $A$ is defined as follows:

$$R^p_{2\text{-}\mathcal{PR}}(A) := p \cdot \sum_{i \in V \setminus A} \mathbf{P}[v_i(A) \geq p].$$

The algorithm works as follows: It fixes an $\varepsilon = o(|V|^{-1})$.

1. For every integer $\rho$, $0 \leq \rho \leq \varepsilon^{-1}$, do:
   (a) Given that the uniform price in the second round is $p = M_p \cdot \rho \cdot \varepsilon$, using the approximation algorithm for non-negative submodular maximization in [4], find the influence set $A_\rho$.
   (b) Let $L_\rho$ be the revenue from giving the item to set $A_\rho$ and setting uniform price to $p = \rho\varepsilon$.
2. Output the set $A_\rho$ and price $\rho\varepsilon$ for which $L_\rho$ is maximized.

In order to prove Theorem 11, we need to prove the following two properties of the revenue function:

1. The revenue function $R^p_{2\text{-}\mathcal{PR}}(A)$ is continuous in $p$ for $0 \leq p \leq M_p$ and for $\delta > 0$,

$$\left| R^{p+\delta}_{2\text{-}\mathcal{PR}}(A) - R^p_{2\text{-}\mathcal{PR}}(A) \right| \leq O(\delta \cdot M_p \cdot |V|) \leq O(\delta) \cdot \widehat{R}^{\mathrm{IE}}_{2\text{-}\mathcal{PR}},$$

   where $\widehat{R}^{\mathrm{IE}}_{2\text{-}\mathcal{PR}}$ is the optimal revenue for any influence set $A$ and uniform price $p$;
2. The function $R^p_{2\text{-}\mathcal{PR}}(A)$ is submodular and non-negative in $A$.

The following lemma shows that if we sample the set of prices with a step $\delta$, we will not lose too much revenue (only revenue proportional to $\delta$).

**Lemma 11.** *Let each client have a valuation function which satisfies the assumptions from Theorem 11. Then for any $\delta > 0$,*

$$\left| R^{p+\delta}_{2\text{-}\mathcal{PR}}(A) - R^p_{2\text{-}\mathcal{PR}}(A) \right| \leq O(\delta \cdot M_p \cdot |V|) \leq O(\delta) \cdot \widehat{R}^{IE}_{2\text{-}\mathcal{PR}}.$$

*Proof.* Note that:

$$\left| R^{p+\delta}_{2\text{-}\mathcal{PR}}(A) - R^p_{2\text{-}\mathcal{PR}}(A) \right| = \left| \sum_{i \in V \setminus A} \left( \mathbf{P}[v_i(A) \geq p+\delta](p+\delta) - \mathbf{P}[v_i(A) \geq p] \cdot p \right) \right| \tag{8}$$

We know that since $v_i(A) = \tilde{v}_i \cdot g_i(A)$,

$$\mathbf{P}[v_i(A) \geq p+\delta] - \mathbf{P}[v_i(A) \geq p] = F_i\left(\frac{p}{g_i(A)}\right) - F_i\left(\frac{p+\delta}{g_i(A)}\right).$$

From the Lagrange mean value theorem, there exists $x$, $\frac{p}{g_i(A)} \leq x \leq \frac{p+\delta}{g_i(A)}$ such that

$$\frac{F_i\left(\frac{p}{g_i(A)}\right) - F_i\left(\frac{p+\delta}{g_i(A)}\right)}{\frac{\delta}{g_i(A)}} = f_i(x) \leq \bar{f}$$

From the above, the Equation 8 and the triangle inequality, we get

$$\left|R^{p+\delta}_{2\text{-}\mathcal{PR}}(A) - R^{p}_{2\text{-}\mathcal{PR}}(A)\right| \leq \frac{1}{g_i(A)} \cdot \left(|V \setminus A| \cdot p\delta\bar{f} + \delta \cdot |V \setminus A|\right) = O(\delta \cdot M_p \cdot |V|); \quad (9)$$

(Recall that $g_i(A) \leq 1$ for all $A \subseteq V$).

Since the probability density function $f_i$ of $\tilde{v}_i$ is non-decreasing on $[0, M_v]$, clearly $\mathbf{P}[\tilde{v}_i \geq \frac{M_v}{2}] \geq \frac{1}{2}$. Since $M_p \leq \mu \cdot M_v$ and $g_i(\emptyset) \geq \mu$, this implies that $\mathbf{P}\left[v_i(\emptyset) \geq \frac{M_p}{2}\right] \geq \frac{1}{2}$. So for the price $p := \frac{M_p}{2}$ and influence set $A := \emptyset$, we get $R^{\frac{M_p}{2}}_{2\text{-}\mathcal{PR}}(\emptyset) \geq \frac{M_p}{2} \cdot |V|$. Therefore, $\widehat{R}^{\text{IE}}_{2\text{-}\mathcal{PR}} = \Omega(M_p \cdot |V|)$, and

$$O(\delta \cdot M_p \cdot |V|) \leq O(\delta) \cdot \widehat{R}^{\text{IE}}_{2\text{-}\mathcal{PR}}.$$

□

In order to prove the submodularity of $R^{p}_{2\text{-}\mathcal{PR}}(A)$, we will need the following lemma:

**Lemma 12.** *Let $f$ be a convex function, let $a \geq \{b, c\} \geq d$ and $(a-b) \geq (c-d)$. Then*

$$f(a) - f(b) \geq f(c) - f(d).$$

*Proof.* We will first show that for any convex function $f$ and $x \geq y \geq z$, the following two claims hold:

$$\frac{f(x) - f(z)}{x - z} \geq \frac{f(y) - f(z)}{y - z}; \quad (10)$$

$$\frac{f(x) - f(y)}{x - y} \geq \frac{f(x) - f(z)}{x - z}. \quad (11)$$

From the convexity of $f$, we have

$$f(y) \leq \frac{y - z}{x - z} f(x) + \frac{x - y}{x - z} f(z). \quad (12)$$

Multiplying both sides with $(x - z)$ and subtracting $f(z)(x - z)$ we get

$$(f(x) - f(z)) \cdot (y - z) \geq (f(y) - f(z)) \cdot (x - z),$$

which is exactly Equation 10.

When we multiply Equation 12 with $(x - z)$ and add $f(x) \cdot (x - y) - f(y) \cdot (x - z) - f(z) \cdot (x - y)$, we get

$$(f(x) - f(y)) \cdot (x - z) \geq (f(x) - f(z)) \cdot (x - y),$$

i.e., the Equation 11.

Now if $a \geq b$, $a \geq c$, $b \geq d$, $c \geq d$, we can use Equations 10 and 11 to show

$$\frac{f(c) - f(d)}{c - d} \leq \frac{f(a) - f(d)}{a - d} \leq \frac{f(a) - f(b)}{a - b}.$$

Since we assume that $(a - b) \geq (c - d)$, we get that

$$f(a) - f(b) \geq f(c) - f(d).$$

□

In our proof of submodularity of $R^p_{2\text{-}\mathcal{PR}}(A)$, we will follow the same scheme as [11]. We will prove the equivalent of their Lemma 1: the individual revenue function of client $i$,

$$h^p_i(A) := \mathbf{P}[v_i(A) \geq p] \cdot p$$

is monotone and submodular in $A$. Applying Lemma 2 of [11], we then get that $R^p_{2\text{-}\mathcal{PR}}(A)$ is submodular.

**Lemma 13.** *Let the price and the individual valuation functions $v_i$ satisfy the assumptions from Theorem 11. Then for each client, its revenue function $h^p_i(A)$ is monotone and submodular in $A$.*

*Proof.* From our assumptions on $v_i(A)$, we can write the revenue function $h^p_i$ as

$$h^p_i(A) = \mathbf{P}[v_i(A) \geq p] \cdot p = \left(1 - F_i\left(\frac{p}{g_i(A)}\right)\right) \cdot p.$$

Clearly $h^p_i(A)$ is monotone, since the function $g_i(A)$ is non-decreasing, and $F_i(x)$ is also non-decreasing. Moreover by the assumptions $M_p \leq M_v \cdot \mu$ and $g_i(\emptyset) \geq \mu$ we have that $0 \leq \frac{p}{g_i(A)} \leq M_v$.

To prove the submodularity of $h^p_i(A)$, we need to show that for any $S \subseteq T$ and any $v \notin T$,

$$h^p_i(T \cup \{v\}) - h^p_i(T) \leq h^p_i(S \cup \{v\}) - h^p_i(S).$$

This is equivalent to

$$F_i\left(\frac{p}{g_i(T)}\right) - F_i\left(\frac{p}{g_i(T \cup \{v\})}\right) \leq F_i\left(\frac{p}{g_i(S)}\right) - F_i\left(\frac{p}{g_i(S \cup \{v\})}\right). \quad (13)$$

We will apply Lemma 12. From our assumption that the second derivative of $F_i$ is non-negative ($f_i$ is differentiable and non-decreasing), we get that $F_i$ is convex. From the fact that $g_i(A)$ is monotone, clearly $\frac{p}{g_i(S)} \geq \frac{p}{g_i(S \cup \{v\})}$, $\frac{p}{g_i(T)} \geq \frac{p}{g_i(T \cup \{v\})}$.

Next we show that also

$$\frac{p}{g_i(T)} - \frac{p}{g_i(T \cup \{v\})} \leq \frac{p}{g_i(S)} - \frac{p}{g_i(S \cup \{v\})}.$$

This is equivalent to

$$\frac{g_i(T \cup \{v\}) - g_i(T)}{g_i(T \cup \{v\}) \cdot g_i(T)} \leq \frac{g_i(S \cup \{v\}) - g_i(S)}{g_i(S \cup \{v\}) \cdot g_i(S)}$$

This holds from the submodularity of $g_i(A)$ and from the fact that $g_i(T \cup \{v\}) \cdot g_i(T) \geq g_i(S \cup \{v\}) \cdot g_i(S)$ since $g_i(A)$ is monotone.

Therefore, we can apply Lemma 12 to prove Equation 13, and the submodularity of $h_i^p(A)$. □

**Proof of Theorem 11.** Let $\hat{p}$ be the price and $\hat{A}$ the influence set which maximizes $R^p_{2\text{-}\mathcal{PR}}(A)$. The algorithm of [11] samples the price space with step $\varepsilon$. Let $\rho$ be such that $\hat{p} \in [\rho\varepsilon, (\rho+1)\varepsilon)$. From Lemma 11 we know that

$$R^{\rho\varepsilon}_{2\text{-}\mathcal{PR}}(\hat{A}) \geq (1-\varepsilon) \cdot R^{\hat{p}}_{2\text{-}\mathcal{PR}}(\hat{A}).$$

From the result of [4] and from the fact that the revenue function $R^p_{2\text{-}\mathcal{PR}}$ is non-negative and submodular (Lemma 13), for the fixed price $p' = \rho\varepsilon$, we can find an influence set $A'_{p'}$ such that

$$R^{p'}_{2\text{-}\mathcal{PR}}(A'_{p'}) \geq \frac{1}{2} \cdot R^{p'}_{2\text{-}\mathcal{PR}}(\hat{A}_{p'}) \geq \frac{1}{2} \cdot (1-\varepsilon) \cdot R^{\hat{p}}_{2\text{-}\mathcal{PR}}(\hat{A}).$$

This concludes the proof of Theorem 11. □