

Welfare Maximization with Friends-of-Friends Network Externalities

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Abstract

Online social networks allow the collection of large amounts of data about the influence between users connected by a friendship-like relationship. When distributing items among agents forming a social network, this information allows us to exploit network externalities that each agent receives from his neighbors that get the same item. In this paper we consider Friends-of-Friends (2-hop) network externalities, i.e., externalities that not only depend on the neighbors that get the same item but also on neighbors of neighbors. For these externalities we study a setting where multiple different items are assigned to unit-demand agents. Specifically, we study the problem of welfare maximization under different types of externality functions. Let n be the number of agents and m be the number of items. Our contributions are the following: (1) We show that welfare maximization is APX-hard; we show that even for step functions with 2-hop (and also with 1-hop) externalities it is NP-hard to approximate social welfare better than $(1 - 1/e)$. (2) On the positive side we present (i) an $O(\sqrt{n})$ -approximation algorithm for general concave externality functions, (ii) an $O(\log m)$ -approximation algorithm for linear externality functions, and (iii) an $(1 - 1/e)^{\frac{1}{6}}$ -approximation algorithm for 2-hop step function externalities. We also improve the result from [6] for 1-hop step function externalities by giving a $(1 - 1/e)/2$ -approximation algorithm.

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1 Introduction

Assume you have to form a committee and need to decide whom to choose as a member. It seems like a good strategy to select members from your network that are well-connected to the whole field so that not only the knowledge of the actual members but also of their whole network can be called upon when needed. Along the same vein assume you want to play a multiplayer online game but you do not have enough friends who are willing to play with you. Then it is a good idea to ask these friends to contact their friends whether they are willing to play as well. Both these settings can be modeled by a social network graph and in both settings not the *direct* (or *1-hop*) neighbors alone, but instead the 1-hop neighbors in combination with the *neighbors of neighbors* (or *2-hop neighbors*) are the decisive factor. Note that the 2-hop neighborhoods cannot be modeled by 1-hop neighborhoods through the insertion of an additional edge (to the neighbor of the neighbor) as we require that every *participating* neighbor of a neighbor is adjacent to a *participating* neighbor. In the above

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example, we can only get the opinion of a contact of a contact if we asked the contact before. In the same way, the participation of a friend of a friend will only be possible if there is a participating friend that invites him.

There has been a large body of work by social scientists and, in the last decade, also by computer scientists (see e.g., the influential paper by Kempe, Kleinberg, and Tardos [22] and its citations) to model and analyze the effect of 1-hop neighborhoods. The study of 2-hop neighborhoods has received much less attention (see e.g., [11, 21]). This is surprising as a recent study [16] of the Facebook network shows that the median Facebook user has 31k people as “friends of friends” and due to some users with very large friend lists, the average friends-of-friends reach in their study is even 156k. Thus, even if each individual friend of a friend has only a small influence on a Facebook user, in aggregate the influence of the friends-of-friends might be large and should not be ignored.

We, therefore, initiate the study of the influence of 2-hop neighborhoods in the popular *assignment* setting, where items are assigned to users whose values for the item depend on who else in their neighborhood has the item. There is a large body of work on mechanisms and pricing strategies for this problem with a single [5, 17, 4, 1, 7, 24, 12, 3, 15] or multiple items [8, 2, 6, 14, 26, 27, 25, 20] when the valuation function of a user depends *solely* on the 1-hop neighborhood of a user and the user itself. All this work assumes that there is an infinite supply of items (of each type if there are different items) and the users have unit-demand, that is want to buy only *one* item. This is frequently the case, for example, if the items model competing products or if the user has to make a binary decision between participating or not participating. In the above examples, this requirement would model that each user can only be in one committee or play one game at a time.

Thus, we study the allocation of items to users in a setting with 2-hop network externalities, where the valuation that a user derives from the products depends on herself, her 1-hop, and her 2-hop neighborhood with the goal of maximizing the social welfare of the allocation. The prior work that is most closely related to our work is the work by Bhalgat et al. [6], where they study the multi-item setting with 1-hop externality functions and give approximation algorithms for different classes of externality functions. For linear externalities they give a $1/64$ -approximation algorithm and for step function externalities they get an approximation ratio of $(1 - 1/e)/16 \approx 0.04$. Additionally they present a $2^{O(d)}$ -approximation algorithm for convex externalities that are bounded by polynomials of degree d and a polylogarithmic approximation algorithm for submodular externalities.

1.1 Our Results

The Model: Let $G = (V, E)$ be an undirected graph modeling the social network. Consider any agent $j \in V$ who receives item $i \in I$, and let $S_{ij} \subseteq V \setminus \{j\}$ denote the (2-hop) *support* of agent j for item i : this is the set of agents who contribute towards the valuation of j . Specifically, an agent $j' \in V \setminus \{j\}$ belongs to the set S_{ij} iff j' gets item i *and* the following condition holds: either j' is a neighbor of j (i.e., $(j, j') \in E$), or j and j' have a common neighbor j'' who also gets item i . The valuation received by agent j is equal to $\lambda_{ij} \cdot \text{ext}_{ij}(|S_{ij}|)$, where λ_{ij} is the agent’s *intrinsic valuation* and $\text{ext}_{ij}(|S_{ij}|)$ is her 2-hop *externality* for item i . The goal is to compute an assignment of items to the agents that maximizes the social welfare, which is defined as the sum of the valuations obtained by the agents.

We study three types of 2-hop externality functions, namely concave, linear and step function externalities.

Step-function externalities: Consider a game requiring a minimal or fixed number of players (larger than two), e.g., Bridge or Canasta, then the externality is a step function. For step functions we show that it is **NP-hard** to approximate the social welfare within a factor of $(1 - 1/e)$. The result holds for 1-hop and 2-hop externalities. We also show that the problem remains **APX-hard** when the number of items is restricted to 2. Then we give an $(1 - 1/e)/6 \approx 0.1$ -approximation algorithm for 2-hop step function externalities. Note that this is within a factor of $1/6$ of the hardness bound. Our technique also leads to a *combinatorial* $(1 - 1/e)/2 \approx 0.3$ -approximation algorithm for 1-hop step function externalities, improving the approximation ratio of the *LP-based* algorithm in [6].

Linear externalities: First we show that social welfare maximization for linear 2-hop externality functions is **NP-hard**.¹ Then we give an $O(\log n)$ -approximation algorithm for linear 2-hop externalities. For these externality functions we can relax the unit-demand requirement. Specifically, we can handle the setting where each user j can buy up to c_j different items, where c_j is a parameter given in the input².

Concave externalities: We give an $O(\sqrt{n})$ -approximation algorithm when the externality functions $ext_{ij}(\cdot)$ are concave and monotone.

Extensions: Our algorithms for linear and concave externalities can be further generalized to allow a weighting of 2-hop neighbors so that 2-hop neighbors have a lower weight than 1-hop neighbors. This can be useful if it is important that the influence of 2-hop neighbors does not completely dominate the influence of the 1-hop neighbors.

Techniques: The main challenge in dealing with 2-hop externalities is as follows. Fix an agent j who gets an item i , and let $V_i \subseteq V$ denote the set of all agents who get item i . Recall that the agent j 's externality is given by $ext_{ij}(|S_{ij}|)$, where the set S_{ij} is called the support of agent j . The problem is that $|S_{ij}|$, as a function of $V_i \setminus \{j\}$, is not submodular. This is in sharp contrast with the 1-hop setting, where the support for the agent's externality comes only from the set of her 1-hop neighbors who receive item i .

All the mechanisms in [6] use the same basic approach: First solve a suitable LP-relaxation and then round its values independently for each item i . In the 2-hop setting, however, the lack of submodularity of the support size (as described above) leads to many dependencies in the rounding step. Nevertheless, we show how to extend the technique in [6] to achieve the approximation algorithm for linear 2-hop externality functions, using a novel LP. We further give a simple combinatorial algorithm with an approximation guarantee of $O(\sqrt{n})$ for 2-hop concave externalities. For this, we show that either an $\Omega(1/\sqrt{n})$ -fraction of the optimal social welfare comes from a single item, or we can reduce our problem to a setting with 1-hop step function externalities by losing an $(1 - \Omega(1/\sqrt{n}))$ -fraction of the objective.

Our approach for 2-hop step functions is different. We use a novel decomposition of the graph into a maximal set of disjoint connected sets of size 3, 2, and 1. We say an assignment is *consistent* if it assigns all the nodes (i.e., users) in the same connected set the same item.

¹ Theorem 3.1 in [6] claims that the welfare maximization problem for linear 1-hop externality functions in complete graphs is **MaxSNP-hard**, which would imply our result, but, as we show in Appendix G, this claim is not true.

² This is also true for the results in [6]. In both results, the assumption is that the valuation functions are additive over the items.

We show first that restricting ourselves to consistent assignments reduces the maximum welfare by at most a factor of $1/6$. Finally, we show that finding the optimal consistent assignment is equal to maximizing social welfare in a scenario where agents are not unit demand, do not influence each other, and have valuation functions that are fractionally subadditive in the items they get assigned. For the latter we use the $(1 - 1/e)$ -approximation algorithm by Feige [10].

2 Notations and Preliminaries

We are given a simple undirected graph $G = (V, E)$ with $|V| = n$ nodes. Each node $j \in V$ in this graph is an agent, and there is an edge $(j, j') \in E$ iff the agents j and j' are friends with each other. There is a set of m items $I = \{1, \dots, m\}$. Each item is available in unlimited supply, and each agent wants to get at most one item. An *assignment* $\mathcal{A} : V \rightarrow I$ specifies the item received by every agent, and under this assignment, $u_j(\mathcal{A}, G)$ gives the *valuation* of an agent $j \in V$. Our goal is to find an assignment that maximizes the *social welfare* $\sum_{j \in V} u_j(\mathcal{A}, G)$, i.e., the sum of the valuations of the agents.

Let $F_j^1(G)$ (resp. $F_j^2(G)$) be the 1-hop (resp. 2-hop) neighborhood of node j .

$$F_j^1(G) = \{j' \in V : (j, j') \in E\}, \quad F_j^2(G) = \bigcup_{j' \in F_j^1(G)} F_{j'}^1(G) \setminus (F_j^1(G) \cup \{j\}).$$

Define $V_i(\mathcal{A}, G) = \{j \in V : \mathcal{A}(j) = i\}$ to be the set of agents who receive item $i \in I$ under the assignment $\mathcal{A}$. Let $N_j^1(i, \mathcal{A}, G) = F_j^1(G) \cap V_i(\mathcal{A}, G)$ denote the set of agents in $F_j^1(G)$ who receive item i under the assignment $\mathcal{A}$. Further, let $N_j^2(i, \mathcal{A}, G) = F_j^2(G) \cap V_i(\mathcal{A}, G) \cap \left(\bigcup_{j'' \in N_j^1(i, \mathcal{A}, G)} F_{j''}^1(G) \right)$ denote the set of agents in $F_j^2(G)$ who receive item i under the assignment $\mathcal{A}$ and are adjacent to some node in $N_j^1(i, \mathcal{A}, G)$.

The *support* of an agent $j \in V$ for item $i \in I$ is defined as $S_{ij}(\mathcal{A}, G) = N_j^1(i, \mathcal{A}, G) \cup N_j^2(i, \mathcal{A}, G)$. This is the set of agents contributing towards the valuation of j for item i . Let λ_{ij} be the *intrinsic valuation* of agent j for item i , and let $ext_{ij}(|S_{ij}(\mathcal{A}, G)|)$ be the *externality* of the agent for the same item. The agent's valuation from the assignment $\mathcal{A}$ is given by the following equality.

$$u_j(\mathcal{A}, G) = \lambda_{\mathcal{A}(j), j} \cdot ext_{\mathcal{A}(j), j}(|S_{\mathcal{A}(j), j}(\mathcal{A}, G)|).$$

We consider three types of externalities in this paper.

► **Definition 1.** In *concave externality* it holds that $ext_{ij}(t)$ is a monotone and concave function of t , with $ext_{ij}(0) = 0$, for every item $i \in I$ and agent $j \in V$.

► **Definition 2.** In *linear externality* it holds that for all $j \in V$, $i \in I$ and every nonnegative integer t , we have $ext_{ij}(t) = t$.

We extend the step function definition of [6] as follows to 2-hop neighborhoods.

► **Definition 3.** For integer $s \geq 1$, in *s-step function externality* it holds that for all $j \in V$, $i \in I$ and every nonnegative integer t , we have $ext_{ij}(t)$ is 1 if $t \geq s$ and 0 otherwise.

We omit the symbol G from these notations if the underlying graph is clear from the context. All the missing proofs from Sections 4, 5 appear in the Appendix.

3 An $O(\sqrt{n})$ -Approximation for Concave Externalities

For the rest of this section, we fix the underlying graph G , and assume that the agents have concave externalities as per Definition 1. We also fix the intrinsic valuations λ_{ij} and the externality functions $ext_{ij}(\cdot)$.

- Let $\mathcal{A}^* \in \arg \max_{\mathcal{A}} \left\{ \sum_{j \in V} u_j(\mathcal{A}) \right\}$ be an assignment that maximizes the social welfare, and let $\text{OPT} = \sum_{j \in V} u_j(\mathcal{A}^*)$ be the optimal social welfare.
- Let $X^* = \{j \in V : |S_{\mathcal{A}^*(j),j}(\mathcal{A}^*)| \geq \sqrt{n}\}$ be the set of agents with support size at least $\sqrt{n}$ under the assignment $\mathcal{A}^*$, and let $Y^* = V \setminus X^*$ be the remaining set of agents.

Since X^* and Y^* partition the set of agents V , there can be two possible cases. Half of the social welfare under $\mathcal{A}^*$ is coming (1) either from the agents in X^* , or (2) from the agents in Y^* . Lemma 4 shows that in the former case there is a *uniform assignment*, where every agent gets the same item, that retrieves $1/(2\sqrt{n})$ -fraction of the optimal social welfare. We consider the latter case in Lemma 5, and reduce it to a problem with 1-hop externalities.

► **Lemma 4.** *If $\sum_{j \in X^*} u_j(\mathcal{A}^*) \geq \text{OPT}/2$, then there is an item $i \in I$ such that $\sum_{j \in V} u_j(\mathcal{A}^i) \geq \text{OPT}/(2\sqrt{n})$, where $\mathcal{A}^i$ is the assignment that gives item i to every agent in V , that is, $\mathcal{A}^i(j) = i$ for all $j \in V$.*

Proof. Define the set of items $I(X^*) = \bigcup_{j \in X^*} \{\mathcal{A}^*(j)\}$.

We claim that $|I(X^*)| \leq \sqrt{n}$. To see why the claim holds, let $V_i^* = \{j \in V : \mathcal{A}^*(j) = i\}$ be the set of agents who receive item i under $\mathcal{A}^*$. Now, fix any item $i \in I(X^*)$, and note that, by definition, there is an agent $j \in X^*$ with $\mathcal{A}^*(j) = i$. Thus, we have $|V_i^*| \geq |S_{ij}(\mathcal{A}^*)| \geq \sqrt{n}$. We conclude that $|V_i^*| \geq \sqrt{n}$ for every item $i \in I(X^*)$. Since $\sum_{i \in I(X^*)} |V_i^*| \leq |V| = n$, it follows that $|I(X^*)| \leq \sqrt{n}$.

To conclude the proof of the lemma, we now make the following observations.

$$\begin{aligned} \sum_{j \in X^*} u_j(\mathcal{A}^*) &= \sum_{i \in I(X^*)} \sum_{j \in X^* : \mathcal{A}^*(j) = i} u_j(\mathcal{A}^*) \leq |I(X^*)| \cdot \max_{i \in I(X^*)} \left(\sum_{j \in X^* : \mathcal{A}^*(j) = i} u_j(\mathcal{A}^*) \right) \\ &\leq \sqrt{n} \cdot \max_{i \in I(X^*)} \left(\sum_{j \in X^* : \mathcal{A}^*(j) = i} u_j(\mathcal{A}^i) \right) \leq \sqrt{n} \cdot \max_{i \in I(X^*)} \left(\sum_{j \in V} u_j(\mathcal{A}^i) \right) \end{aligned}$$

The lemma holds since $\text{OPT}/(2\sqrt{n}) \leq \sum_{j \in X^*} u_j(\mathcal{A}^*)/\sqrt{n} \leq \max_{i \in I(X^*)} \left(\sum_{j \in V} u_j(\mathcal{A}^i) \right)$. ◀

For every item $i \in I$ and agent $j \in V$, we now define the externality function $\hat{ext}_{ij}(t)$ and the valuation function $\hat{u}_j(\mathcal{A})$.

$$\hat{ext}_{ij}(t) = \begin{cases} ext_{ij}(1) & \text{if } t \geq 1; \\ 0 & \text{otherwise.} \end{cases} \quad \hat{u}_j(\mathcal{A}) = \lambda_{\mathcal{A}(j),j} \cdot \hat{ext}_{ij}(|N_j^1(i, \mathcal{A})|) \quad (1)$$

Clearly, for every assignment $\mathcal{A} : V \rightarrow I$, we have $0 \leq \sum_{j \in V} \hat{u}_j(\mathcal{A}) \leq \sum_{j \in V} u_j(\mathcal{A})$. Also note that the valuation function $\hat{u}_j(\cdot)$ depends only on the 1-hop neighborhood of the agent j . Specifically, if an agent j gets an item i , then her valuation is $\lambda_{ij} \cdot ext_{ij}(1)$ if at least one of her 1-hop neighbors also gets the same item i , and zero otherwise. Bhalgat et al. [6] gave an LP-based $O(1)$ -approximation for finding an assignment $\mathcal{A} : V \rightarrow I$ that maximizes the social welfare in this setting (also see Section 5 for a combinatorial algorithm). In the lemma below, we show that if the agents in Y^* contribute sufficiently towards OPT under the assignment $\mathcal{A}^*$, then by losing an $O(\sqrt{n})$ -factor in the objective, we can reduce our original problem to the one where the externalities are $\hat{ext}_{ij}(\cdot)$ and the valuations are $\hat{u}_j(\cdot)$.

► **Lemma 5.** *If $\sum_{j \in Y^*} u_j(\mathcal{A}^*) \geq \text{OPT}/2$, then $\sum_{j \in V} \hat{u}_j(\mathcal{A}^*) \geq \text{OPT}/(2\sqrt{n})$.*

Proof. Consider a node $j \in Y^*$ that makes nonzero contribution towards the objective (i.e., $u_j(\mathcal{A}^*) > 0$) and suppose that it gets items i (i.e., $\mathcal{A}^*(j) = i$). Since $u_j(\mathcal{A}^*) > 0$, we have $S_{ij}(\mathcal{A}^*) = N_j^1(i, \mathcal{A}^*) \cup N_j^2(i, \mathcal{A}^*) \neq \emptyset$, which in turn implies that $N_j^1(i, \mathcal{A}^*) \neq \emptyset$. Thus, we have $\hat{u}_j(\mathcal{A}^*) = \lambda_{ij} \cdot \text{ext}_{ij}(1)$. Since $|S_{ij}(\mathcal{A}^*)| \leq \sqrt{n}$ and $\text{ext}_{ij}(\cdot)$ is a concave function, we have $\text{ext}_{ij}(1) \geq \text{ext}_{ij}(|S_{ij}(\mathcal{A}^*)|)/|S_{ij}(\mathcal{A}^*)| \geq \text{ext}_{ij}(|S_{ij}(\mathcal{A}^*)|)/\sqrt{n}$. Multiplying both sides of this inequality by λ_{ij} , we conclude that $\hat{u}_j(\mathcal{A}^*) \geq u_j(\mathcal{A}^*)/\sqrt{n}$ for all agents $j \in Y^*$ with $u_j(\mathcal{A}^*) > 0$. In contrast, if $u_j(\mathcal{A}^*) = 0$, then the inequality $\hat{u}_j(\mathcal{A}^*) \geq u_j(\mathcal{A}^*)/\sqrt{n}$ is trivially true. Thus, summing over all $j \in Y^*$, we infer that $\sum_{j \in Y^*} \hat{u}_j(\mathcal{A}^*, G) \geq \sum_{j \in Y^*} u_j(\mathcal{A}^*, G)/\sqrt{n} \geq \text{OPT}/(2\sqrt{n})$. The lemma now follows since $\sum_{j \in V} \hat{u}_j(\mathcal{A}^*, G) \geq \sum_{j \in Y^*} \hat{u}_j(\mathcal{A}^*, G)$. ◀

The algorithm for concave externalities. We run two procedures. Procedure (1) returns an assignment $\mathcal{A}' \in \arg \max_{i \in I} \left(\sum_{j \in V} u_j(\mathcal{A}^i) \right)$, where $\mathcal{A}^i(j) = i$ for all $i \in I$ and $j \in V$. Procedure (2) invokes the algorithm in [6] and returns an assignment $\mathcal{A}''$ such that $\sum_{j \in V} \hat{u}_j(\mathcal{A}'') \geq (1/\alpha) \cdot \max_{\mathcal{A}} \left(\sum_{j \in V} \hat{u}_j(\mathcal{A}) \right)$ for some constant $\alpha \geq 1$, where the function $\hat{u}_j(\cdot)$ is defined as in equation 1. Our algorithm now compares these two assignments $\mathcal{A}'$ and $\mathcal{A}''$ and returns the one that gives maximum social welfare, i.e, we output an assignment $\mathcal{A}''' \in \arg \max_{\mathcal{A} \in \{\mathcal{A}', \mathcal{A}''\}} \left(\sum_{j \in V} u_j(\mathcal{A}) \right)$.

► **Theorem 6.** *The algorithm described above gives an $O(\sqrt{n})$ -approximation for social welfare under 2-hop, concave externalities.*

Proof. Recall the notations introduced in the beginning of Section 3. Since the set of agents V is partitioned into $X^* \subseteq V$ and $Y^* = V \setminus X^*$, either $\sum_{j \in X^*} u_j(\mathcal{A}^*) \geq \text{OPT}/2$ or $\sum_{j \in Y^*} u_j(\mathcal{A}^*) \geq \text{OPT}/2$. In the former case, Lemma 4 guarantees that $\sum_{j \in \mathcal{A}'''} u_j(\mathcal{A}''') \geq \sum_{j \in \mathcal{A}'} u_j(\mathcal{A}') \geq \text{OPT}/(2\sqrt{n})$. In the latter case, by Lemma 5 we have $\sum_{j \in \mathcal{A}'''} u_j(\mathcal{A}''') \geq \sum_{j \in \mathcal{A}''} u_j(\mathcal{A}'') \geq \sum_{j \in \mathcal{A}''} \hat{u}_j(\mathcal{A}'') \geq \sum_{j \in \mathcal{A}^*} \hat{u}_j(\mathcal{A}^*)/\alpha \geq \text{OPT}/(2\alpha\sqrt{n})$. Since α is a constant, we conclude that the social welfare returned by our algorithm is always within an $O(\sqrt{n})$ -factor of the optimal social welfare. ◀

4 An $O(\log m)$ -Approximation for Linear Externalities

In this section, we assume that the input graph $G = (V, E)$ is of the following form. The set V is partitioned into three groups V_1, V_2 and V_3 . Further, an edge in E either connects a node in V_1 with a node in V_2 , or connects a node in V_2 with a node in V_3 . Our goal is to assign the items to the agents in such a way as to maximize the social welfare from the set V_1 . We refer to this problem as RESTRICTED-WELFARE, and prove Theorem 7 in Appendix A.

► **Theorem 7.** *Any α -approximation algorithm for the RESTRICTED-WELFARE problem can be converted into an $O(\alpha)$ -approximation algorithm for the welfare-maximization problem in general graphs with linear (or even concave) externalities.*

Consider the LP below. Here, the variable $\alpha(i, j, k)$ indicates if both the agents $j \in V_1$ and $k \in F_j^1$ received item $i \in I$. If this variable is set to one, then agent j gets one unit of externality from agent k . Similarly, the variable $\beta(i, j, l)$ indicates if both the agents $j \in V_1, l \in V_3 \cap F_j^2$ received item $i \in I$ and there is at least one agent $k \in F_j^1 \cap F_l^1$ who also received the same item. If this variable is set to one, then agent j gets one unit of externality from agent l . Clearly, the total valuation of agent j for item i is given by $\sum_{k \in V_2 \cap F_j^1} \lambda_{ij} \cdot \alpha(i, j, k) + \sum_{l \in V_3 \cap F_j^2} \lambda_{ij} \cdot \beta(i, j, l)$. Summing over all the items and all the agents in V_1 , we see that the LP-objective encodes the social welfare of the set V_1 .

$$\text{Maximize: } \sum_{j \in V_1} \sum_{i \in I} \lambda_{ij} \cdot \left(\sum_{k \in V_2 \cap F_j^1} \alpha(i, j, k) + \sum_{l \in V_3 \cap F_j^2} \beta(i, j, l) \right) \quad (2)$$

$$\beta(i, j, l) \leq \min\{w(i, l), y(i, j)\} \quad \forall i \in I, j \in V_1, l \in V_3 \cap F_j^2 \quad (3)$$

$$\beta(i, j, l) \leq \sum_{k \in F_j^1 \cap F_l^1} z(i, k) \quad \forall i \in I, j \in V_1, l \in V_3 \cap F_j^2 \quad (4)$$

$$\alpha(i, j, k) \leq \min\{y(i, j), z(i, k)\} \quad \forall i \in I, j \in V_1, k \in V_2 \cap F_j^1 \quad (5)$$

$$\sum_i y(i, j) \leq 1, \sum_i z(i, k) \leq 1, \sum_i w(i, l) \leq 1 \quad \forall j, k, l \quad (6)$$

$$0 \leq z(i, k), y(i, j), w(i, l), \alpha(i, j, k), \beta(i, j, l) \quad \forall i, j, k, l \quad (7)$$

The variables $y(i, j)$, $z(i, k)$ and $w(i, l)$ respectively indicate if an agent $j \in V_1$, $k \in V_2$, $l \in V_3$ received item $i \in I$. Constraints 6 state that an agent can get at most one item. Constraint 5 says that if $\alpha(i, j, k) = 1$, then both $y(i, j)$ and $z(i, k)$ must also be equal to one. Constraint 3 states that if $\beta(i, j, l) = 1$, then both $y(i, j)$ and $w(i, l)$ must also be equal to one. Finally, note that if an agent $l \in V_3$ contributes one unit of externality to an agent $j \in V_1$ for an item $i \in I$, then there must be some agent $k \in F_j^1 \cap F_l^1$ in V_2 who received the same item. This condition is encoded in constraint 4. Thus, we have the following lemma.

► **Lemma 8.** *The LP is a valid relaxation of the RESTRICTED-WELFARE problem.*

Before proceeding towards the rounding scheme, we perform a preprocessing step as described in the next lemma, whose proof appears in Appendix B.

► **Lemma 9.** *In polynomial time, we can get a feasible solution to the LP that gives an $O(\log m)$ approximation to the optimal objective, and ensures that each $\alpha(i, j, k), \beta(i, j, l), y(i, j), w(i, l) \in \{0, \gamma\}$ for some real number $\gamma \in [0, 1]$, and that each $z(i, k) \leq \gamma$.*

We now present the rounding scheme for LP (see Algorithm 1). Here, the set W_i denotes the set of agents that have not yet been assigned any item when the rounding scheme enters the FOR loop for item i (see Step 2). Note that the sets T_i might overlap, but these conflicts are resolved in Line 9 by intersecting T_i with W_i , which is disjoint with all previous $T_j, j < i$.

Algorithm 1 Rounding Scheme for LP

1. In accordance with Lemma 9, compute a feasible solution to the LP.
Set $T_0 \leftarrow \emptyset$, and $W_0 \leftarrow V = V_1 \cup V_2 \cup V_3$.
 2. FOR all items $i \in I = \{1, \dots, m\}$:
 3. Set $W_i \leftarrow W_{i-1} \setminus T_{i-1}$, and $T_i \leftarrow \emptyset$.
 4. Pick a value η_i uniformly at random from $[0, 1]$.
 5. IF $\eta_i \leq \gamma$:
 6. FOR all nodes $j \in V_1$:
IF $y(i, j) = \gamma$, then with probability $1/4$, set $T_i \leftarrow T_i \cup \{j\}$.
 7. FOR all nodes $l \in V_3$:
IF $w(i, l) = \gamma$, then with probability $1/4$, set $T_i \leftarrow T_i \cup \{l\}$.
 8. FOR all nodes $k \in V_2$:
With probability $z(i, k)/(4\gamma)$, set $T_i \leftarrow T_i \cup \{k\}$.
 9. Assign item i to all nodes in $W_i \cap T_i$, i.e., set $\mathcal{A}(t) \leftarrow i$ for all $t \in W_i \cap T_i$.
 10. RETURN the (random) assignment $\mathcal{A}$.
-

► **Lemma 10.** *For all $t \in V$ and all $i \in I$, we have $\mathbf{P}[t \in W_i] \geq 3/4$. Thus, $\mathbf{P}[\{t_1, t_2, t_3\} \subseteq W_i] \geq 1/4$ for all $t_1, t_2, t_3 \in V$.*

Proof. We will prove the lemma for a node in V_1 , the argument extends to $V_2 \cup V_3$.

Fix any node $j \in V_1$ and any item $i \in I$, and consider an indicator random variable $\Gamma_{i'j}$ that is set to one iff $j \in T_{i'}$. It is easy to check that $\mathbf{E}[\Gamma_{i'j}] = y(i', j)/4$ for all items $i' \in I$. By constraint 6 and linearity of expectation, we thus have: $\mathbf{E}[\sum_{i' < i} \Gamma_{i'j}] = \sum_{i' < i} y(i', j)/4 \leq 1/4$. Applying Markov's inequality, we get $\mathbf{P}[\sum_{i' < i} \Gamma_{i'j} = 0] \geq 3/4$. In other words, with probability at least $3/4$, we have that $j \notin T_{i'}$ for all $i' < i$. Under this event, we must have $j \in W_i$.

We have $\mathbf{P}[t \notin W_i] \leq 1/4$ for all $t \in \{t_1, t_2, t_3\}$. $\mathbf{P}[\{t_1, t_2, t_3\} \subseteq W_i] \geq 1/4$ now follows from applying union-bound over these three events. $\blacktriangleleft$

In the first step, when we find a feasible solution to the LP in accordance with Lemma 9, we lose a factor of $O(\log m)$ in the objective. Below, we will show that the remaining steps in the rounding scheme result in a loss of at most a constant factor in the approximation ratio.

For all items $i \in I$, nodes $j \in V_1$, and nodes $k \in F_j^1$, $l \in F_j^2$, we define the random variables $X(i, j, k)$ and $Y(i, j, l)$. Their values are determined by the outcome $\mathcal{A}$ of our randomized rounding. To be more specific, we have that $X(i, j, k) = 1$ if both j and k receive item i , and $X(i, j, k) = 0$ otherwise. Further, $Y(i, j, l) = 1$ if both j and l receive item i and there is some node in $F_j^1 \cap F_l^1$ that also received item i , and $Y(i, j, l) = 0$ otherwise. Now, the valuation of any agent $j \in V_1$ from the (random) assignment $\mathcal{A}$ is:

$$u_j(\mathcal{A}) = \sum_{i \in I} \left(\sum_{k \in F_j^1} \lambda_{ij} \cdot X(i, j, k) + \sum_{l \in F_j^2} \lambda_{ij} \cdot Y(i, j, l) \right) \quad (8)$$

We will analyze the expected contribution of the rounding scheme to each term in the LP-objective. Towards this end, we prove the following lemmas. The proof of Lemma 11 appears in Appendix C.

► **Lemma 11.** *For all $i \in I, j \in V_1, k \in F_j^1$, we have $\mathbf{E}_{\mathcal{A}}[X(i, j, k)] \geq \delta \cdot \alpha(i, j, k)$, where $\delta > 0$ is a sufficiently small constant.*

► **Lemma 12.** *For all $i \in I, j \in V_1, l \in F_j^2$, we have $\mathbf{E}_{\mathcal{A}}[Y(i, j, l)] \geq \delta \cdot \beta(i, j, l)$, where δ is a sufficiently small constant.*

Proof. Fix an item $i \in I$, a node $j \in V_1$ and a node $l \in F_j^2$. If $\beta(i, j, l) = 0$ the lemma is trivially true. Otherwise suppose for the rest of the proof that $\beta(i, j, l) = y(i, j) = w(i, l) = \gamma$.

Let $\mathcal{E}_i$ be the event that $\eta_i \leq \gamma$ (see Step 4 in Algorithm 1). Let $Z(i, k)$ be an indicator random variable that is set to one iff node $k \in V_2$ is included in the set T_i by our rounding scheme (see Step 8 in Algorithm 1). We have:

$$\mathbf{P}[\mathcal{E}_i] = \gamma, \text{ and } \mathbf{P}[Z(i, k) = 1 \mid \mathcal{E}_i] = z(i, k)/4\gamma \text{ for all } k \in V_2 \quad (9)$$

Thus, conditioned on the event $\mathcal{E}_i$, the expected number of common neighbors of j and l who are included in the set T_i is given by

$$\mu_i := \mathbf{E} \left[\sum_{k \in F_j^1 \cap F_l^1} Z(i, k) \mid \mathcal{E}_i \right] = \sum_{k \in F_j^1 \cap F_l^1} z(i, k)/4\gamma \geq 1/4 \quad (10)$$

Note that conditioned on the event $\mathcal{E}_i$, the random variables $Z(i, k)$ are mutually independent. Thus, applying Chernoff bound on Equation 10, we infer that with constant probability, at

least one common neighbor of j and l will be included in the set T_i . To be more precise, define $T_{i,j,l} = T_i \cap F_j^1 \cap F_l^1$. For some sufficiently small constant δ_1 , we have:

$$\mathbf{P}[T_{i,j,l} \neq \emptyset \mid \mathcal{E}_i] = \mathbf{P}\left[\sum_{k \in F_j^1 \cap F_l^1} Z(i, k) > 0 \mid \mathcal{E}_i\right] \geq 1 - e^{-1/8} = \delta_1 \quad (11)$$

Let $\mathcal{E}_{i,j,l}$ be the event that the following two conditions hold simultaneously: (a) $T_{i,j,l} \neq \emptyset$, AND (b) j, l , and an arbitrary node from $T_{i,j,l}$ —each of these three nodes is included in W_i . Now, Equation 11 and Lemma 10 imply that $\mathbf{P}[\mathcal{E}_{i,j,l} \mid \mathcal{E}_i] \geq \delta_2$ for $\delta_2 = \delta_1/4$. Putting all these observations together, we obtain that $\mathbf{P}[Y(i, j, l) = 1] = \mathbf{P}[\mathcal{E}_i] \cdot \mathbf{P}[\mathcal{E}_{i,j,l} \mid \mathcal{E}_i] \cdot \mathbf{P}[j, l \in T_i \mid \mathcal{E}_{i,j,l} \cap \mathcal{E}_i] = \gamma \cdot \delta_2 \cdot (1/4) \cdot (1/4) = \delta \cdot \gamma = \delta \cdot \beta(i, j, l)$ for $\delta = \delta_2/16$. ◀

► **Theorem 13.** *The rounding scheme in Algorithm 1 gives an $O(\log m)$ -approximation to the RESTRICTED-WELFARE problem.*

Proof. In the first step, when we find a feasible solution to the LP in accordance with Lemma 9, we lose a factor of $O(\log m)$ in the objective. At the end of the remaining steps, the expected valuation of an agent $j \in V_1$ is given by:

$$\begin{aligned} \mathbf{E}_{\mathcal{A}}[u_j(\mathcal{A})] &= \sum_{i \in I} \lambda_{ij} \cdot \left(\sum_{k \in F_j^1} \mathbf{E}_{\mathcal{A}}[X(i, j, k)] + \sum_{l \in F_l^1} \mathbf{E}_{\mathcal{A}}[Y(i, j, l)] \right) \\ &= \Theta \left(\sum_{i \in I} \lambda_{ij} \cdot \left(\sum_{k \in F_j^1} \alpha(i, j, k) + \sum_{l \in F_l^1} \beta(i, j, l) \right) \right) \end{aligned}$$

The first equality follows from linearity of expectation, while the second equality follows from Lemma 11 and Lemma 12. Thus, the expected valuation of any agent in V_1 is within a constant factor of the fractional valuation of the same agent under the feasible solution to the LP obtained at the end of Step 1 (see Algorithm 1). Summing over all the agents in V_1 , we get the theorem. ◀

We can generalize the above approach to the following setting: Each user j is given an integer c_j and can be assigned up to c_j different items (each at most once). For this we replace for each item i and node j the constraint $\sum_i y(i, j) \leq 1$ by the two constraints $\sum_i y(i, j) \leq c_j$ and $y(i, j) \leq 1$ and adapt the proof of Lemma 10.

We give the modified proof in Appendix D.

Finally, we show NP-hardness for linear externalities, not only in the 2-hop setting but also for 1-hop. The proof is provided in Appendix G.³

► **Theorem 14.** *Maximizing social welfare under linear externalities is NP-hard.*

5 Constant Approximation for Step Function Externalities

In this section, our goal is to maximize the social welfare when the agents have general step function externalities, i.e., to receive externality an agent needs a certain number of

³ As mentioned before Theorem 3.1 in [6] claims that the welfare maximization problem for linear 1-hop externality functions in complete graphs is MaxSNP-hard, which would imply our result, but, as we show in Appendix G, this claim is not true.

1- and 2-hop neighbors having the same product. We will show that no constant factor approximation is possible unless a bound on the number of neighbors an agent needs to receive externality is given. Thus we consider the case of 2-step function externalities, where only two neighbors are needed (see Definition 3) and give a $\frac{1}{6} \cdot (1 - 1/e)$ -approximation algorithm for this problem. Notice that if we consider step functions that just require one neighbor the problem reduces to the 1-hop step function scenario in [6]. However, our algorithm gives a $\frac{1}{2} \cdot (1 - 1/e)$ -approximation for this scenario improving the result in [6].

In the following we assume 2-step function externalities. Let $G_{V'}$ denote the subgraph induced by $V' \subseteq V$. For the rest of this section, the term “triple” will refer to any (unordered) set of three nodes $T = \{j_1, j_2, j_3\}$ such that G_T is connected. Similarly, the term “pair” will refer to any (unordered) set of two nodes $\{j_1, j_2\}$ that are connected by an edge in E .

We first compute a maximal collection of mutually disjoint triples in the graph G . We denote this collection by $\mathcal{T}$, and let $V(\mathcal{T}) = \bigcup_{T \in \mathcal{T}} T \subseteq V$. The graph $G_{V \setminus V(\mathcal{T})}$, by definition, consists of a mutually disjoint collection of pairs (say $\mathcal{P}$) and a set of isolated nodes (say B). We thus have the following lemma.

► **Lemma 15.** *In $G = (V, E)$, there is no edge that connects a node $j \in B$ with another node in B or with a node belonging to a pair in $\mathcal{P}$. Furthermore, there is no edge that connects two nodes j, j' belonging to two different pairs $P, P' \in \mathcal{P}$.*

► **Definition 16.** An assignment $\mathcal{A}$ is *consistent* iff two agents get the same item whenever they belong to the same triple or the same pair. To be more specific, for all $j, j' \in V$, we have that $\mathcal{A}(j) = \mathcal{A}(j')$ if either (a) $j, j' \in T$ for some triple $T \in \mathcal{T}$ or (b) $\{j, j'\} \in \mathcal{P}$.

The next lemma shows that by losing a factor of 6 in the approximation ratio, we can focus on maximizing the social welfare via a consistent assignment.

► **Lemma 17.** *The social welfare from the optimal consistent assignment is at least $(1/6) \cdot \text{OPT}$, where OPT is the maximum social welfare over all assignments.*

Proof. Let $\mathcal{A}^*$ be an assignment (not necessarily consistent) that gives maximum social welfare. We convert it into a (random) consistent assignment $\mathcal{A}$ as follows. For each triple $\{j_1, j_2, j_3\} \in \mathcal{T}$, we pick one of the items $\mathcal{A}^*(j_1), \mathcal{A}^*(j_2), \mathcal{A}^*(j_3)$ uniformly at random, and assign that item to all the three agents j_1, j_2, j_3 . Similarly, for each pair $\{j_1, j_2\} \in \mathcal{P}$, we pick one of the items $\mathcal{A}^*(j_1), \mathcal{A}^*(j_2)$ uniformly at random, and assign that item to both the agents j_1, j_2 . The events corresponding to different triples and pairs are mutually independent. Finally, the remaining agents (those who are in B) get the same items as in $\mathcal{A}^*$. It is easy to see that the resulting assignment $\mathcal{A}$ is consistent. We claim that $\mathbf{E}[u_j(\mathcal{A})] \geq (1/6) \cdot u_j(\mathcal{A}^*)$ for all $j \in V$. To prove this claim, we consider three cases.

Case 1 ($j \in B$): Let $\mathcal{A}^*(j) = i$. Since $j \in B$, it always gets the same item under $\mathcal{A}$, i.e., $\mathcal{A}(j) = i$. Now, if $u_j(\mathcal{A}^*) = 0$, then the claim is trivially true. Otherwise it must be the case that $\mathcal{A}^*(j') = i$ for some neighbor j' of j . Since $j \in B$, this neighbor j' must be part of some triple $T \in \mathcal{T}$ (see Lemma 15). With probability at least $1/3$ all the three nodes in T are assigned item i under $\mathcal{A}$ and at least two nodes of T are in the 2-hop neighborhood of j . In that event j gets the same valuation as in $\mathcal{A}^*$, and we have that $\mathbf{E}[u_j(\mathcal{A})] \geq (1/3) \cdot u_j(\mathcal{A}^*)$.

Case 2 (j belongs to a pair in $\mathcal{P}$): Consider the pair $P = \{j, j'\} \in \mathcal{P}$, which has j and another node (say j') as its members. Let $\mathcal{A}^*(j) = i$. As in Case 1, if $u_j(\mathcal{A}^*) = 0$, then the claim is trivially true. Otherwise it must be the case that there exists a node j'' with $\mathcal{A}^*(j'') = i$ such that j'' is either a neighbor of j or a neighbor of j' . Since $\{j, j'\} \in \mathcal{P}$, this agent j'' must be part of some triple $T \in \mathcal{T}$ (see Lemma 15). Let $\mathcal{E}_1$ be the event that all the three

nodes in T are assigned item i under $\mathcal{A}$. Similarly, let $\mathcal{E}_2$ be the event that both the nodes $j, j' \in P$ get the same item i under $\mathcal{A}$. Since these two events are mutually independent, we have that $\mathbf{P}[\mathcal{E}_1 \cap \mathcal{E}_2] \geq (1/3) \cdot (1/2) = 1/6$, and in the event $\mathcal{E}_1 \cap \mathcal{E}_2$, we have $u_j(\mathcal{A}) = u_j(\mathcal{A}^*)$. It follows that $\mathbf{E}[u_j(\mathcal{A})] \geq (1/6) \cdot u_j(\mathcal{A}^*)$.

Case 3 (j belongs to a triple in $\mathcal{T}$): Consider the triple $T = \{j, j', j''\} \in \mathcal{T}$ which has, besides j , two other nodes (say j' and j'') as its members. With probability at least $1/3$, all these three nodes are assigned item $\mathcal{A}^*(j)$ under $\mathcal{A}$, and in this event we have $u_j(\mathcal{A}) \geq u_j(\mathcal{A}^*)$. It follows that $\mathbf{E}[u_j(\mathcal{A})] \geq (1/3) \cdot u_j(\mathcal{A}^*)$.

Now, we take a sum of the inequalities $\mathbf{E}[u_j(\mathcal{A})] \geq (1/6) \cdot u_j(\mathcal{A}^*)$ over all agents $j \in V$, and by linearity of expectation infer that the expected social welfare under the consistent assignment $\mathcal{A}$ is within a factor of 6 of the optimal social welfare. This concludes the proof of the lemma. $\blacktriangleleft$

Next, we will give an $(1 - 1/e)$ -approximation algorithm for finding a consistent assignment of items that maximizes the social welfare. Along with Lemma 17, this will imply the main result of this section (see Theorem 20).

We use the term “resource” to refer to either a pair $P \in \mathcal{P}$ or an agent $j \in B$. Let $\mathcal{R} = \mathcal{P} \cup B$ denote the set of all resources. We say that a resource $r \in \mathcal{R}$ *neighbors* a triple $T \in \mathcal{T}$ iff in the graph $G = (V, E)$ either (a) $r = \{j, j'\} \in \mathcal{P}$ and some node in $\{j, j'\}$ is adjacent to some node in T , or (b) $r = j \in B$ and j is adjacent to some node in T . We slightly abuse the notation (see Section 2) and let $N(T) \subseteq \mathcal{R}$ denote the set of resources that are neighbors of $T \in \mathcal{T}$.

By definition, every consistent assignment ensures that if two agents belong to the same triple in $\mathcal{T}$ (resp. the same pair in $\mathcal{P}$), then both of them get the same item. We say that *the item is assigned to a triple (resp. resource)*. Note that the triples do not need externality from outside. To be more specific, the contribution of a triple $T \in \mathcal{T}$ to the social welfare is always equal to $\sum_{j \in T} \lambda_{i,j}$, where i is the item assigned to T . Resources, however, do need outside externality, which by Lemma 15 can come only from a triple in $\mathcal{T}$.

► Lemma 18. *In a consistent assignment, if a resource $r \in \mathcal{R}$ makes a positive contribution to the social welfare, then it neighbors some triple $T_r \in \mathcal{T}$, and both the resource r and the triple T_r receive the same item.*

Proof. If a resource contributes a nonzero amount to the social welfare, then it must receive nonzero externality from the assignment. By Lemma 15, such externality can come only from a triple in $\mathcal{T}$. The lemma follows. $\blacktriangleleft$

Thus, given a consistent assignment $\mathcal{A}$ consider the following mapping $T_{\mathcal{A}}(r)$ of a resource $r \in \mathcal{R}$ to triples in $\mathcal{T}$ in accordance with Lemma 18: If the resource r makes zero contribution towards the social welfare (a case not covered by the lemma), then we let $T_{\mathcal{A}}(r)$ be any arbitrary triple from $\mathcal{T}$. Otherwise $T_{\mathcal{A}}(r)$ denotes an (arbitrary) neighboring triple of $\mathcal{T}$ that receives the same item as r . We say that the triple $T_{\mathcal{A}}(r)$ *claims* the resource r .

For ease of exposition, let $\lambda_{i,r}(T)$ be the valuation of the resource r when both the resource and the triple T that claims it get item $i \in I$, i.e.,

$$\lambda_{i,r}(T) = \begin{cases} \lambda_{i,j} + \lambda_{i,j'} & \text{if } r = \{j, j'\} \in \mathcal{P} \text{ and } r \in N(T); \\ \lambda_{i,j} & \text{if } r = j \in B \text{ and } r \in N(T); \\ 0 & \text{if } r \notin N(T). \end{cases}$$

Now, any consistent assignment $\mathcal{A}$ can be interpreted as follows. Under such an assignment, every triple $T \in \mathcal{T}$ claims the subset of the resources $S_T = \{r \in \mathcal{R} : T_{\mathcal{A}}(r) = T\}$; the subsets corresponding to different triples being mutually exclusive. A triple T and the resources in S_T all get the same item (say $i \in I$). The valuation obtained from them is $u_T(S_T, i) = \sum_{j \in T} \lambda_{i,j} + \sum_{r \in S_T} \lambda_{i,r}(T)$.

If our goal is to maximize the social welfare, then, naturally, for every triple T , we will pick the item that maximizes $u_T(S_T, i)$, thereby extracting a valuation of $u_T(S_T) = \max_i u_T(S_T, i)$. The next lemma shows that this function is fractionally subadditive.

► **Lemma 19.** *The function $u_T(S_T)$ is fractionally subadditive in S_T .*

The preceding discussion shows that the problem of computing a consistent assignment for welfare maximization is equivalent to the following setting. We have a collection of triples $\mathcal{T}$, and a set of resources $\mathcal{R}$. We will distribute these resources amongst the triples, i.e., every triple T will get a subset $S_T \subseteq \mathcal{R}$, and these subsets will be mutually exclusive. The goal is to maximize the sum $\sum_{T \in \mathcal{T}} u_T(S_T)$, where the functions $u_T(\cdot)$'s are fractionally subadditive. By a celebrated result of Feige [10], we can get an $(1 - 1/e)$ -approximation algorithm for this problem if we can implement the following subroutine (called *demand oracle*) in polynomial time: Each resource r is given a “cost” $p(r)$ and we need to determine for each triple T a set of resources S_T^* that maximizes $u_T(S_T) - \sum_{r \in S_T} p(r)$ over all sets S_T . Such a demand oracle can be implemented in polynomial time using a simple greedy algorithm for each T and each item i : Add a resource r to S_T^* iff $\lambda_{i,r}(T) > p(r)$. The result of the approximation algorithm assigns each triple T a subset S_T and we then pick the item i that maximizes $u_T(S_T, i)$ over all items i . Together with Lemma 15, this implies the theorem stated below.

► **Theorem 20.** *We can get a polynomial-time $\frac{1}{6} \cdot (1 - 1/e)$ -approximation algorithm for the problem of maximizing social welfare under 2-step function externalities.*

The algorithm can be easily adapted for 1-hop step function externalities. The difference being that instead of computing a maximal collection $\mathcal{T}$ of mutually disjoint triples, one computes a maximal collection of mutually disjoint pairs.

► **Theorem 21.** *We can get a polynomial-time $\frac{1}{2} \cdot (1 - 1/e)$ -approximation algorithm for maximizing social welfare under 1-step function externalities.*

Finally, we present our hardness results for step functions. By a reduction from MAX INDEPENDENT SET (see Appendix I) we can show that, for unbounded s , there is no constant factor approximation. The main idea is that we modify the graph such that we replace each edge by a path of length three and each of the original nodes j wants a different item, while j can only get positive externalities when having a support of $2\delta_j$ (δ_j the node degree of j). The valuations of the newly introduced nodes are set to 0. That is, nodes that are adjacent in the original graph have two common neighbors in the constructed graph, want different items, need all their neighbors as support, and thus only one of them can have positive valuation.

► **Theorem 22.** *For any $\varepsilon > 0$ the problem of maximizing social welfare under arbitrary s -step function externalities is not approximable within $O(n^{1/4-\varepsilon})$ unless $\text{NP} = \text{P}$, and not approximable within $O(n^{1/2-\varepsilon})$ unless $\text{NP} = \text{ZPP}$.*

Second, we show that maximizing social welfare under 2-step function externalities is APX-hard and thus no PTAS can exist. This is by a reduction from MAX COVERAGE. The APX-hardness for two items is by a reduction from SAT.

► **Theorem 23.** *The problem of maximizing social welfare under step function externalities is APX-hard, in particular, there is no polytime $1 - \frac{1}{e} + \epsilon$ -approximation algorithm (unless $P = NP$). Furthermore, the problem remains APX-hard even if there are only two items.*

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A

 Proof of Theorem 7

Theorem 7. *Any α -approximation algorithm for the RESTRICTED-WELFARE problem can be converted into an $O(\alpha)$ -approximation algorithm for the welfare-maximization problem in general graphs with linear externalities.*

We want to maximize the social welfare in a general graph $G = (V, E)$ under linear externality functions. First, we randomly partition the set of nodes V into three groups: V_1, V_2, V_3 . To be more precise, we iterate over all the nodes in the graph. While considering a specific node $j \in V$ we randomly add it to one of the sets V_1, V_2 , or V_3 , and we ensure that these random events corresponding to the different nodes in V are mutually independent. Next, we delete all the edges that join a node in V_1 with a node in V_3 , and all the edges whose both endpoints lie within the same subset $V_t, t \in \{1, 2, 3\}$. Let this new (random) tripartite graph be $G^* = (V = V_1 \cup V_2 \cup V_3, E^*)$. Finally, given any assignment $\mathcal{A}$ of the items to the nodes in this graph, we define the *restricted social welfare* to be the sum of the valuations obtained by the agents in V_1 . Thus, we ignore the contributions of the nodes in $V_2 \cup V_3$. These nodes only serve the purpose of building up the externalities of the nodes in V_1 . A node in V_2 can only be a 1-hop neighbor of a node in V_1 , whereas a node in V_3 can only be a 2-hop neighbor of a node in V_1 .

Theorem 7 will follow from Lemma 24 and Lemma 25.

► **Lemma 24.** *The social welfare from an assignment $\mathcal{A} : V \rightarrow I$ in the input graph $G = (V, E)$ is at least the restricted social welfare from the same assignment $\mathcal{A}$ in the graph $G^* = (V, E^*)$.*

Proof. Follows from the facts that $E \supseteq E^*$ and that in restricted social welfare we only consider the valuations of the nodes in $V_1 \subseteq V$. ◀

► **Lemma 25.** *The social welfare from any assignment in the graph G is within a constant factor of the expected value of the restricted social welfare from the same assignment in the random graph G^* .*

Proof. Throughout the proof of the lemma, fix any assignment $\mathcal{A} : V \rightarrow I$, and let $SW(\mathcal{A}, G)$ (resp. $SW^*(\mathcal{A}, G^*)$) denote the social welfare (resp. restricted social welfare) from $\mathcal{A}$ in the graph G (resp. G^*). We will show $\mathbf{E}[SW^*(\mathcal{A}, G^*)] = \delta \cdot SW(\mathcal{A}, G)$ for some constant $\delta > 0$.

Fix any node $j \in V$ and let $i \in I$ be the item assigned to it. By definition:

$$u_j(\mathcal{A}, G) = \lambda_{ij} \cdot (|N_j^1(i, \mathcal{A}, G)| + |N_j^2(i, \mathcal{A}, G)|) \quad (12)$$

For every node $k \in N_j^1(i, \mathcal{A}, G)$ (resp. $l \in N_j^2(i, \mathcal{A}, G)$) we define an indicator random variable X_k (resp. Y_l) as follows.

$$X_k = \begin{cases} 1 & \text{if } j \in V_1 \text{ and } k \in V_2; \\ 0 & \text{otherwise.} \end{cases}$$

$$Y_l = \begin{cases} 1 & \text{if } j \in V_1 \text{ and } l \in V_3, \text{ and at least one node} \\ & \text{from } N_j^1(i, \mathcal{A}, G) \cap N_l^1(i, \mathcal{A}, G) \text{ is included in } V_2; \\ 0 & \text{otherwise.} \end{cases}$$

Next, we define a random variable W_j as follows.

$$W_j = \begin{cases} u_j(\mathcal{A}, G^*) & \text{if } j \in V_1; \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$W_j \geq \sum_{k \in N_j^1(i, \mathcal{A}, G)} \lambda_{ij} \cdot X_k + \sum_{l \in N_j^2(i, \mathcal{A}, G)} \lambda_{ij} \cdot Y_l \quad (13)$$

Since the nodes in V are partitioned into the subsets V_1, V_2, V_3 uniformly and independently at random, we have:

$$\mathbf{E}[X_k] = \mathbf{P}[X_k = 1] = (1/3) \cdot (1/3) = 1/9 \quad \forall k \in N_j^1(i, \mathcal{A}, G) \quad (14)$$

$$\mathbf{E}[Y_l] = \mathbf{P}[Y_l = 1] \geq (1/3) \cdot (1/3) \cdot (1/3) = 1/27 \quad \forall l \in N_j^2(i, \mathcal{A}, G) \quad (15)$$

From equations 13, 14 15, and linearity of expectation, it follows that

$$\mathbf{E}[W_j] \geq \delta \cdot \lambda_{ij} \cdot \{|N_j^1(i, \mathcal{A}, G)| + |N_j^2(i, \mathcal{A}, G)|\} \text{ for some constant } \delta > 0. \quad (16)$$

Equation 12 and 16 imply that

$$\mathbf{E}[W_j] \geq \delta \cdot u_j(\mathcal{A}, G) \quad (17)$$

Now, the expected value of the restricted social welfare from $\mathcal{A}$ in G^* is:

$$\mathbf{E}[SW^*(\mathcal{A}, G^*)] = \sum_{j \in V} \mathbf{E}[W_j] \geq \delta \cdot \sum_{j \in V} u_j(\mathcal{A}, G) = \delta \cdot SW(\mathcal{A}, G)$$

This concludes the proof of the lemma. ◀

B Proof of Lemma 9

Lemma 9. *In polynomial time, we can get a feasible solution to the LP that gives an $O(\log m)$ approximation to the optimal objective, and ensures that each $\alpha(i, j, k), \beta(i, j, l), y(i, j), w(i, l) \in \{0, \gamma\}$ for some real number $\gamma \in [0, 1]$, and that each $z(i, k) \leq \gamma$.*

Proof. We compute the optimal solution to LP 2, and partition the $\alpha(i, j, k)$ and $\beta(i, j, l)$ variables into two groups (*large* and *small*) depending on whether they are greater than or less than $1/m^2$. By losing at most a $1/m$ fraction of the objective, we can set all the small variables to zero. To see this, suppose that the claim is false, i.e., the contribution of these small variables exceeds $1/m$ fraction of the total objective. Then we can scale up all these small variables by a factor of m , set all the large $\alpha(i, j, k), \beta(i, j, l)$'s to zero, and set every $z(i, k), y(i, j), w(i, l)$ to $1/m$. This will satisfy all the constraints, and the total contribution towards the objective by the erstwhile small variables will get multiplied by m , which, in turn, will imply that their new contribution actually exceeds the optimal objective. Thus, we reach a contradiction.

We discretize the range $[1/m^2, 1]$ in powers of two, thereby creating $O(\log m)$ intervals, and accordingly, we partition the large variables into $O(\log m)$ groups. The variables in the same group are within a factor 2 of each other. By losing an $O(\log m)$ factor in the approximation ratio, we select the group that contributes the most towards the LP-objective. Let all the variables in this group lie in the range $[\gamma, 2\gamma]$. We now make the following transformation. All the $\alpha(i, j, k), \beta(i, j, l)$'s in this group are set to γ . This way we lose another factor of at most 2 in the LP-objective. All the remaining $\alpha(i, j, k), \beta(i, j, l)$'s are set to zero. At this stage, we have ensured that each $\alpha(i, j, k), \beta(i, j, l) \in \{0, \gamma\}$.

Now, it is easy to check that if a $y(i, j)$ or a $w(i, l)$ is set to a value less than γ , then it plays no role whatsoever in ensuring the feasibility of the LP-solution. This holds as constraints 3, 4 ensure that any $\alpha(i, j, k)$ or $\beta(i, j, l)$ corresponding to those variables must be set to zero (recall that each $\alpha(i, j, k), \beta(i, j, l)$ is either γ or zero). Thus, we set any $y(i, j), w(i, l) < \gamma$ to zero, and we set the remaining $y(i, j), w(i, l)$'s to γ . Since each $\alpha(i, j, k), \beta(i, j, l) \in \{0, \gamma\}$, the latter transformation does not violate the feasibility of the solution. Finally, we set $z(i, k) \leftarrow \min(z(i, k), \gamma)$ for each $i \in I, k \in V_2$. ◀

Note that after the processing according to Lemma 9 there might exist i, j, k , and l such that $z(i, k)$ and $\beta(i, j, l)$ both equal γ , but $\alpha(i, j, k)$ equals 0. In this case we could increase the value of the solution by setting $\alpha(i, j, k)$ to γ but this is not necessary for our rounding procedure as it ignores both the α as well as the β values.

C

 Proof of Lemma 11

Lemma 11. For all $i \in I, j \in V_1, k \in F_j^1$, we have $\mathbf{E}_{\mathcal{A}}[X(i, j, k)] \geq \delta \cdot \alpha(i, j, k)$, where $\delta > 0$ is a sufficiently small constant.

Proof. Fix an item $i \in I$, a node $j \in V_1$ and a node $k \in F_j^1$. If $\alpha(i, j, k) = 0$ the lemma is trivially true. Otherwise suppose for the rest of the proof that $\alpha(i, j, k) = y(i, j) = z(i, k) = \gamma$. Now, we have:

$$\begin{aligned}
 \mathbf{E}_{\mathcal{A}}[X(i, j, k)] &= \mathbf{P}[X(i, j, k) = 1] \\
 &= \mathbf{P}[j, k \in W_i] \cdot \mathbf{P}[\eta_i \leq \gamma \mid j, k \in W_i] \cdot \mathbf{P}[j, k \in T_i \mid (\eta_i \leq \gamma) \wedge (j, k \in W_i)] \\
 &\geq 1/2 \cdot \gamma \cdot (1/4) \cdot (z(i, k)/4\gamma) \\
 &= \delta \cdot \gamma = \delta \cdot \alpha(i, j, k) \quad (\text{where } \delta = 1/32)
 \end{aligned}$$

The last but one equality holds since $z(i, k) = \gamma$. ◀

D Multi-Demand Settings

We show how to adapt the proof of Lemma 10 to the more general setting.

► **Lemma 26.** *For every node $t \in V$ and every $i \in I$, we have $\mathbf{P}[t \in W_i] \geq 3/4$.*

Proof. We will prove the lemma for a node in V_1 , the argument being exactly similar for a node in $V_2 \cup V_3$.

Fix any node $j \in V_1$ and any item $i \in I$, and consider an indicator random variable $\Gamma_{i'j}$ that is set to one iff $j \in T_{i'}$. It is easy to check that $\mathbf{E}[\Gamma_{i'j}] = y(i', j)/4$ for all items $i' \in I$. By constraint 6 and linearity of expectation, we thus have: $\mathbf{E}[\sum_{i' < i} \Gamma_{i'j}] = \sum_{i' < i} y(i', j)/4 \leq c_j/4$. Applying Markov's inequality, we get $\mathbf{P}[\sum_{i' < i} \Gamma_{i'j} < c_j] \geq 3/4$. In other words, with probability at least $3/4$, we have that less than c_j many items have been assigned to j when item i is processed. Under this event, we must have $j \in W_i$. The lemma follows. ◀

E Proof of Lemma 19

Lemma 19. *The function $u_T(S_T)$ is fractionally subadditive in S_T .*

Proof. Note that the function $u_T(S_T, i)$ is linear in S_T , for every item $i \in I$. The lemma follows from the fact that the maximum of a set of linear functions is fractionally sub-additive. ◀

F p -hop Externalities

In this Section we generalize our notation for the p -hop case. For $p \geq 2$, let the p -hop neighborhood F_j^p be defined as follows.

$$F_j^p = \bigcup_{j' \in F_j^{p-1}} (F_{j'}^1) \setminus (\bigcup_{q < p} F_j^q \cup \{j\}).$$

and let the p -neighborhood $F_j^{\leq p}$ be $\bigcup_{q \leq p} F_j^q$.

Then the p -hop support $N_j^p(i, \mathcal{A})$ is defined as

$$N_j^p(i, \mathcal{A}) = F_j^p(G) \cap V_i(\mathcal{A}) \cap \left(\bigcup_{j' \in N_j^{p-1}(i, \mathcal{A})} F_{j'}^1(G) \right)$$

and $N_j^{\leq p}(i, \mathcal{A}) = \bigcup_{q \leq p} N_j^q(i, \mathcal{A})$.

Finally the p -support of an agent $j \in V$ for item $i \in I$ is then defined as $S_{ij}^p(\mathcal{A}, G) = |N_j^{\leq p}(i, \mathcal{A}, G)|$

G Hardness for Linear Externalities

Bhalgat et al. [6] state that “*The welfare maximization problem for linear externality and complete graphs is MaxSNP-hard*”. As on complete graphs there is no difference between 1-hop, 2-hop and in general p -hop linear externalities this would imply MaxSNP hardness for our setting. Unfortunately, as we explain next, their claim is not true. There is a quite simple method to find the optimal assignment in the case of a complete graph. Consider only the assignments that pick one item $i \in I$ and assign it to every agent, and select the optimal assignment among them. This can be done in linear time in the number of items m and the

number of agents n . Lemma 27 shows that such an assignment is also optimal among all possible assignments. Assuming $\text{NP} \neq \text{P}$ the incorrectness of Theorem 3.1 in [6] follows.

► **Lemma 27.** *Given convex functions $f_{i,j} : \mathbb{N}_0 \rightarrow \mathbb{R}_{\geq 0}$ with $f_{i,j}(0) = 0$ for all $i \in I := \{1, \dots, m\}$ and all $j \in V$, and a partition of V into m sets $S_1, \dots, S_m$ (i.e., $\bigcup_{i \in I} S_i = V$ and $S_i \cap S_{i'} = \emptyset$ for all $i \neq i' \in I$) it holds that $\sum_{i \in I} \sum_{j \in S_i} f_{i,j}(|S_i|) \leq \max_{i \in I} \sum_{j \in V} f_{i,j}(|V|)$.*

Proof. For the proof we set $n = |V|$ and define $i^* \in \arg \max_{i \in I} \sum_{j \in V} f_{i,j}(|V|)$.

$$\sum_{i \in I} \sum_{j \in S_i} f_{i,j}(|S_i|) \leq \sum_{i \in I} \sum_{j \in S_i} f_{i,j}(n) \frac{|S_i|}{n} \quad (18)$$

$$= \sum_{i \in I} \frac{|S_i|}{n} \sum_{j \in S_i} f_{i,j}(n) \quad (19)$$

$$\leq \sum_{i \in I} \frac{|S_i|}{n} \sum_{j \in V} f_{i,j}(n) \quad (20)$$

$$\leq \sum_{i \in I} \frac{|S_i|}{n} \sum_{j \in V} f_{i^*,j}(n) \quad (21)$$

$$= \sum_{j \in V} f_{i^*,j}(n) \sum_{i \in I} \frac{|S_i|}{n} = \sum_{j \in V} f_{i^*,j}(n) \quad (22)$$

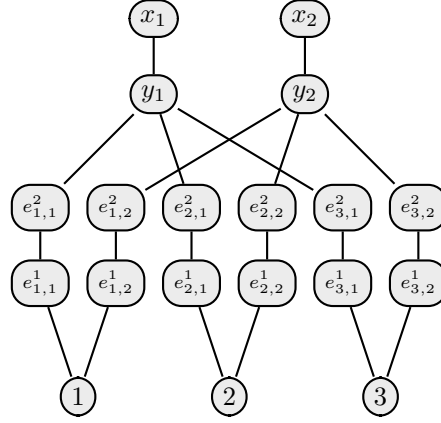
Equality 18 follows from the convexity of $f_{i,j}(\cdot)$ and $f_{i,j}(0) = 0$. Equality 19 holds because $\frac{|S_i|}{n}$ does not depend on j . Inequality 20 follows from the non-negativity of $f_{i,j}(\cdot)$. For Inequality 21 observe that for any fixed $i \in I$ it holds that $\sum_{j \in V} f_{i,j}(n) \leq \sum_{j \in V} f_{i^*,j}(n)$ by the definition of i^* . Further, $\sum_{j \in V} f_{i^*,j}(n)$ does not depend on i , which implies the left-hand side of Equality 22. For the right-hand side of Equality 22 recall that $S_1, \dots, S_m$ is a partition of V , and thus, $\sum_{i \in I} |S_i| = |V| = n$. This gives $\sum_{j \in V} f_{i^*,j}(n)$ which is by definition $\max_{i \in I} \sum_{j \in V} f_{i,j}(|V|)$. ◀

Notice that the valuation of an agent with linear externality is indeed a convex function $f_{i,j}$ with $f_{i,j}(0) = 0$ and in a complete graph the externality of a node only depends on the number of nodes having the same item. Moreover if no one has the item the valuation an agent gets from this item is clearly 0.⁴ Thus the above lemma implies that there is an optimal assignment assigning the same item to all nodes. From that the following theorem follows immediately.

► **Theorem 28.** *The welfare maximization problem for linear externalities and complete graphs can be solved in polynomial time.*

However, on general graphs the problem becomes NP-hard. We next show this NP-hardness in a generalized version of Theorem 14. That is we consider p -hop linear externalities ($p \geq 1$), i.e. the *support* of an agent $j \in V$ for item $i \in I$ be defined as $S_{i,j}^p(\mathcal{A}, G) = |N_j^{\leq p}(i, \mathcal{A}, G)|$ (see also Appendix F). To this end we consider a simpler version of MAX COVERAGE (where all weights w_i are set to 1) which is still NP-hard [9, 23].

⁴ Notice that all these observations hold for both, our definition of linear externalities and the slightly different definition of linear externalities in [6].



■ **Figure 1** An illustration of the graph constructed in the above reduction for $p = 3$, $D = \{1, 2, 3\}$ and $k = 2$.

► **Definition 29.** The input to the MAX COVERAGE problem is a set of domain elements $D = \{1, 2, \dots, n\}$, a collection $\mathcal{S} = \{E_1, \dots, E_m\}$ of subsets of D and a positive integer k . The goal is to find a collection $\mathcal{S}' \subseteq \mathcal{S}$ of cardinality k maximizing $|\bigcup_{E \in \mathcal{S}'} E|$.

Theorem 14 *The problem of maximizing social welfare under p -hop linear function externalities is NP-hard.*

Proof. The proof is by a reduction from the NP-hard problem MAX COVERAGE (see Def. 29). The reduction maps an instance $(D, \mathcal{S}, k)$ of the MAX COVERAGE problem to our problem as follows. It constructs a graph G with a node set V containing $X = \{x_1, \dots, x_k\}$, $Y = \{y_1, \dots, y_k\}$, D and $\{e_{i,j}^h : i \in D, 1 \leq j \leq k, 1 \leq h < p\}$. The undirected edges are given by

- $\{(x_l, y_l) : 1 \leq l \leq k\}$;
- $\{(e_{i,j}^h, e_{i,j}^{h+1}) : i \in D, 1 \leq j \leq k, 1 \leq h < p-1\}$; and
- If $p > 1$ we have edges $\{(i, e_{i,j}^1), (e_{i,j}^{p-1}, y_j) : i \in D, 1 \leq j \leq k\}$.
Otherwise if $p = 1$ then there are edges $\{(i, y_j) : i \in D, 1 \leq j \leq k\}$.

The items are given by $\{E_{i,l} : E_i \in \mathcal{S}, 1 \leq l \leq k\}$. Finally the intrinsic values of an agent $j \in D$ for getting item $E_{i,l}$ are given by

$$\lambda_{E_{i,l},j} = \begin{cases} 1 & \text{if } j \in E_i \\ 0 & \text{otherwise.} \end{cases}$$

while the intrinsic values of and agent x_j are given by

$$\lambda_{E_{i,l},x_j} = \begin{cases} n \cdot k \cdot p & \text{if } l = j \\ 0 & \text{otherwise.} \end{cases}$$

and for all other agents j and items i the weight $\lambda_{i,j} = 0$.

► **Lemma 30.** *The above reduction maps each instance of the MAX COVERAGE problem to a network of agents with linear function externalities in p -neighborhoods such that for the value opt_{MC} of the optimal selection for the MAX COVERAGE instance and opt_{SW} the social welfare of the optimal item assignment it holds that $\text{opt}_{SW} = k^2 np(1 + (p-1)n) + p \cdot \text{opt}_{MC}$.*

Proof. (1) We first show that $opt_{SW} \geq k^2 np(1 + (p-1)n) + p \cdot opt_{MC}$: W.l.o.g., we assume that an optimal coverage Opt_{MC} is given by $\{S_1, \dots, S_k\}$. Consider the following assignment $\mathcal{A}$. For each $1 \leq j \leq k$, the nodes x_j, y_j and $\{e_{i,j}^h : i \in D, 1 \leq h < p\}$ are assigned the item $E_{j,j}$. If a node $d \in D$ is covered by Opt_{MC} , i.e., it is contained in some $S_l \in Opt_{MC}$, then d is assigned the item $E_{l,l}$ (if there are several such S_l pick the one with the lowest index). Otherwise, if d is not covered by Opt_{MC} then item $E_{1,1}$ is assigned to d . Now consider the social welfare of $\mathcal{A}$. First of all, by construction, only the nodes in X and D contribute social welfare. A node $x \in X$ has a p -neighborhood of $1 + (p-1) \cdot n$ many nodes, and all of his p -neighbors get the same item. Thus, x has a valuation of $n \cdot k \cdot p \cdot (1 + (p-1) \cdot n)$. As $|X| = k$ the set X has a social welfare of $n \cdot k^2 \cdot p \cdot (1 + (p-1) \cdot n)$. Now consider $d \in D$. If d is covered by an $S_l \in Opt_{MC}$, i.e., $d \in E_l$, then it was assigned the item $E_{l,l}$. Moreover, $E_{l,l}$ is also assigned to the nodes $\{e_{d,l}^h : 1 \leq h < p\}$ and y_l which are in the p -neighborhood of d . Thus each covered $d \in D$ has valuation $\geq p$ and therefore $opt_{SW} \geq k^2 np(1 + (p-1)n) + p \cdot opt_{MC}$.

(2) It remains to show that $opt_{SW} \leq k^2 np(1 + (p-1)n) + p \cdot opt_{MC}$: We first show that in an optimal assignment Opt_{SW} each $x_l \in X$ must have valuation $n \cdot k \cdot (1 + (p-1) \cdot n)$, i.e., x_l is assigned to the one of the items $\{E_{i,l} : E_i \in \mathcal{S}\}$, and $\{e_{i,j}^h : i \in D, 1 \leq h < p\}$ and y_l are assigned to the same item. Towards a contradiction assume that there is an assignment $\mathcal{A}$ with one $x \in X$ that has lower valuation. Then the social welfare of X in $\mathcal{A}$ is bounded by $nk^2 p(1 + (p-1)n) - nkp$. Each $d \in D$ has a p -neighborhood of size $k \cdot p$. Thus the social welfare of D in $\mathcal{A}$ is bounded by $n \cdot k \cdot p$. We have that the social welfare of $\mathcal{A}$ is bounded by $nk^2 p(1 + (p-1)n)$. But from (1) we know that $opt_{SW} \geq k^2 np(1 + (p-1)n) + p \cdot opt_{MC}$ where $opt_{MC} \geq 1$ for all non trivial instances. We obtain the desired contradiction. Thus we know that in Opt_{SW} the social welfare of X is given by $n \cdot k^2 \cdot (1 + (p-1) \cdot n)$.

Now consider a $i \in D$. The p -neighborhood of i consists connected sets $\{e_{i,j}^h : 1 \leq h \leq p-1\} \cup \{y_j\}$. By the above observation in opt_{SW} all nodes in such set are assigned the same item but all sets are assigned different items (as they belong to different x_j). Hence for $i \in D$ the valuation is either p if it gets an item $E_{i,l}$ with $d \in E_i$ and one of its neighbor sets gets the same item. Otherwise d has valuation 0. We obtain that $opt_{SW} = k^2 np(1 + (p-1)n) + p \cdot |\{d \in D : d \text{ has positive valuation in } opt_{SW}\}|$. Finally, we can construct an k -covering by choosing the sets S_i corresponding to the items assigned to X . By the construction, this k -covering covers all $d \in D$ with positive valuation and thus $opt_{MC} \geq |\{d \in D : d \text{ has positive valuation in } opt_{SW}\}|$. Hence we obtain $opt_{SW} \leq k^2 np(1 + (p-1)n) + p \cdot opt_{MC}$. ◀

By the above lemma, the presented reduction maps each MAX COVERAGE instance to an instance of our problem. Moreover it can be also performed in polynomial time and thus Theorem 14 follows from the corresponding result for MAX COVERAGE [9, 23]. ◀

H Generalization of Step-Function Externalities

We can generalize the result in Section 5 as follows (Recall the definitions from Appendix F).

► **Definition 31.** In an s -step externality function in a p -neighborhood for all $j \in V, i \in I$ and $\mathcal{A} : V \rightarrow I$ it holds that $u_j(\mathcal{A}, G) = \lambda_{\mathcal{A}(j),j} \cdot ext_{\mathcal{A}(j),j}(|S_{\mathcal{A}(j),j}^p(\mathcal{A}, G)|)$ with $ext_{ij}(t)$ is 1 if $t \geq s$ and 0 otherwise.

Note that we can assume $p \leq s$ as the p -hop neighborhood cannot be reached with fewer than p nodes. We show next how to generalize the algorithm of the previous setting to s -step externality functions in s neighborhoods. We leave it as an open question to give an approximation algorithm for s -step externality functions in p -neighborhoods for arbitrary $p \leq s$. Moreover the algorithm can be also applied to scenarios where different nodes j have different s_j -step functions. Then the parameter s is set to the maximum of all s_j . The generalized algorithm works as follows: Analogous to triples in the previous section we compute a maximal collection of mutually disjoint connected sets of size $s + 1$ in the graph and decompose the remaining graph into maximal collections of connected sets of size $s, \dots, 1$. The definition of a consistent assignment carries directly over to this new decomposition of the graph and Lemma 17 can be generalized as follows.

► **Lemma 32.** *The social welfare from the optimal consistent assignment is at least $\frac{1}{(s+1)s} \cdot \text{OPT}$, where OPT is the maximum social welfare over all assignments under s -step function externalities in a s -neighborhood.*

Proof. Let $\mathcal{A}^*$ be an assignment (not necessarily consistent) that gives maximum social welfare. We convert it into a (random) consistent assignment $\mathcal{A}$ as follows. For each of the connected components $\{j_1, \dots, j_l\}$, we pick one of the items $\mathcal{A}^*(j_1), \dots, \mathcal{A}^*(j_l)$ uniformly at random, and assign that item to all the agents in the component. We claim that $\mathbf{E}[u_j(\mathcal{A})] \geq (\frac{1}{(s+1)s}) \cdot u_j(\mathcal{A}^*)$ for all $j \in V$. To prove this claim, we consider two cases.

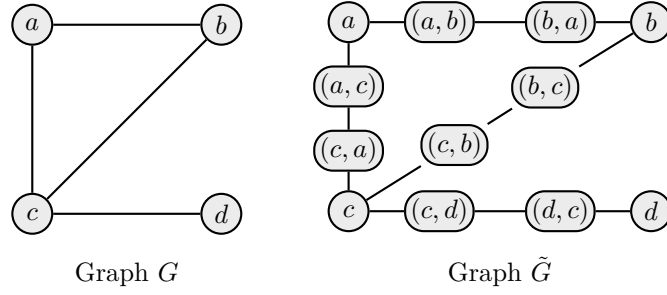
1. (j belongs to a component of size $s + 1$): Consider a component $\{j, j_2, \dots, j_{s+1}\}$ which has, besides j , s other nodes as its members. With probability at least $1/(s + 1)$, all these nodes are assigned item $\mathcal{A}^*(j)$ under $\mathcal{A}$, and in this event we have $u_j(\mathcal{A}) \geq u_j(\mathcal{A}^*)$. It follows that $\mathbf{E}[u_j(\mathcal{A})] \geq (1/(s + 1)) \cdot u_j(\mathcal{A}^*)$.
2. (j belongs to a component of size $\leq s$): Consider a component $P = \{j, j_2, \dots, j_l\}$ which has, besides j , $l - 1$ other nodes as its members. Let $\mathcal{A}^*(j) = i$. As in Case 1, if $u_j(\mathcal{A}^*) = 0$, then the claim is trivially true. Otherwise it must be the case that there exists a node j' with $\mathcal{A}^*(j') = i$ such that j' is either a neighbor of j or a neighbor of one of the $j_2, \dots, j_l$. By construction this agent j must be part of some component T of size s . Let $\mathcal{E}_1$ be the event that all nodes in T are assigned item i under $\mathcal{A}$. Similarly, let $\mathcal{E}_2$ be the event that all the nodes in P get the same item i under $\mathcal{A}$. Since these two events are mutually independent, we have that $\mathbf{P}[\mathcal{E}_1 \cap \mathcal{E}_2] \geq (1/(s + 1)) \cdot (1/l) \geq 1/((s + 1)s)$. In the event $\mathcal{E}_1 \cap \mathcal{E}_2$, we have $u_j(\mathcal{A}) = u_j(\mathcal{A}^*)$ and therefore that $\mathbf{E}[u_j(\mathcal{A})] \geq 1/((s + 1)s) \cdot u_j(\mathcal{A}^*)$.

Now, we take a sum of the inequalities $\mathbf{E}[u_j(\mathcal{A})] \geq (\frac{1}{(s+1)s}) \cdot u_j(\mathcal{A}^*)$ over all agents $j \in V$, and by linearity of expectation infer that the expected social welfare under the consistent assignment $\mathcal{A}$ is within a factor of $1/((s + 1)s)$ of the optimal social welfare. This concludes the proof of the lemma. ◀

The remaining arguments carry through without any changes.

► **Theorem 33.** *We have a polynomial-time $\frac{1}{(s+1)s} \cdot (1 - 1/e)$ -approximation algorithm for maximizing social welfare under s -step function externalities in a s -neighborhood.*

For $s = 1$ this leads to a $\frac{1}{2} \cdot (1 - 1/e)$ -approximation algorithm, improving the $\frac{1}{16} \cdot (1 - 1/e)$ -approximation algorithm of [6].



■ **Figure 2** An illustration of the reduction in the proof of Theorem 22.

I Hardness of Approximation

Theorem 22 *For any $\varepsilon > 0$ the problem of maximizing social welfare under arbitrary s -step function externalities is not approximable within $O(n^{1/4-\varepsilon})$ unless $\text{NP} = \mathcal{P}$, and not approximable within $O(n^{1/2-\varepsilon})$ unless $\text{NP} = \text{ZPP}$.*

Proof. The proof is by an approximation ratio preserving reduction from the MAX INDEPENDENT SET problem defined below.

► **Definition 34.** The input to the MAX INDEPENDENT SET problem is a (undirected) graph $G = (V, E)$. The goal is to find an independent set, i.e. a set $S \subseteq V$ such that no two vertices in S are incident to the same edge, of maximal cardinality.

Now consider the following reduction from the MAX INDEPENDENT SET instance $G = (V, E)$ to our problem. We first compute the degree δ_j of each node $j \in V$. We then construct the graph $\tilde{G} = (\tilde{V}, \tilde{E})$ as follows:

$$\begin{aligned}\tilde{V} &= V \cup \{(v, v'), (v', v) : (v, v') \in E\} \\ \tilde{E} &= \{(v, (v, v')), ((v, v'), (v', v)), ((v', v), v') : (v, v') \in E\}\end{aligned}$$

The set of items is given by $I = \{i_j : j \in V\}$. and the valuations u_j for $j \in V$ are given by:

$$u_j(\mathcal{A}, \tilde{G}) = \begin{cases} 1 & \text{if } \mathcal{A}(j) = i_j \text{ and } |N_j^1(i, \mathcal{A}, \tilde{G}) \cup N_j^2(i, \mathcal{A}, \tilde{G})| \geq 2\delta_j \\ 0 & \text{otherwise} \end{cases}$$

That is each agent $j \in V$ just wants one item, the item i_j and only if at least $2\delta_j$ 1- or 2-hop neighbors in $\tilde{G}$ have the same product. Also for each item there is just one agent that can get revenue from it. For nodes $j \in \tilde{V} \setminus v$ the valuation $u_j(\mathcal{A}, \tilde{G})$ is always zero. Moreover, by construction each node $j \in V$ has exactly $2\delta_j$ 1- and 2-hop neighbors in $\tilde{G}$, i.e. for an agent j to get a positive externality all his 1- and 2-hop neighbors have to get the same item. The construction is illustrated in Figure 2.

► **Lemma 35.** *The above reduction maps each instance of MAX INDEPENDENT SET to a network of agents with step function externalities in 2-neighborhoods such that the cardinality of the maximum independent set is equal to the social welfare of the optimal item assignment.*

Proof. We will show that (a) given an independent set of size k we can construct an item assignment with social welfare $\geq k$ and (b) given an item assignment with social welfare k we can construct an independent set of size $\geq k$.

(a) Given an independent set $S \subseteq V$ we define $\mathcal{A}_S$ as follows. Each $j \in S$ and all of its 1- or 2-hop neighbors get item i_j assigned. $\mathcal{A}$ is well-defined as nodes in $j, j' \in S$ are not adjacent in G and thus have a distance of at least 6 in G . Hence no node of $\tilde{V}$ has more than one item assigned. Now, by construction, for each node $j \in V$ we have $u_j(\mathcal{A}, \tilde{G}) = 1$ and thus the social welfare is at least $|V| = k$.

(b) Given an item assignment $\mathcal{A}$ we social welfare k , we consider the set S of nodes $j \in \tilde{V}$ that have positive utility, i.e. $S = \{j \in V : u_j(\mathcal{A}, \tilde{G}) = 1\}$. We used that only nodes in V can have positive utility and that the utility of a node is either 0 or 1. As the social welfare is the sum over the utilities we have that $|S| = k$. It remains to show that S is an independent set. Suppose that not, then there are $j, j' \in S$ with $(j, j') \in E$. But that means that in $\tilde{G}$ the nodes (j, j') and (j', j) are neighbors of both j and j' . As j has utility 1 it must be assigned item i_j and as its threshold is $2\delta_j$ also all its neighbors must be assigned item i_j , in particular $(j, j'), (j', j)$ must be assigned item i_j . A symmetric argument for j' results that $(j, j'), (j', j)$ must be assigned item $i_{j'}$. As $\mathcal{A}$ can only assign one item to each node this yields or desired contradiction. Hence S is an independent set of size k . ◀

By the above lemma, the presented reduction is approximation ratio preserving and, it can be also performed in polynomial time. We know that MAX INDEPENDENT SET is not approximable within $|V|^{1/2-\epsilon}$ unless $\mathcal{P} = \text{NP}$ [18] and not approximable within $|V|^{1-\epsilon}$ unless $\text{ZPP} = \text{NP}$ [18]. By the above we get the same ratio for our problem but in terms of $|V|$ and not $|\tilde{V}|$. Now as $n = |\tilde{V}| = |V| + 2|E| \leq 3|V|^2$ we get that our problem is not approximable within $O(n^{1/4-\epsilon})$, and $O(n^{1/2-\epsilon})$ respectively. ◀

Here we restate and prove Theorem 23 for the more general s -step externality functions in a p -neighborhood (for the formal definitions see Appendix H). This results immediately imply the statement in Section 5 as in 2-step externality functions in a 2-neighborhood correspond to the step function externalities from Definition 3.

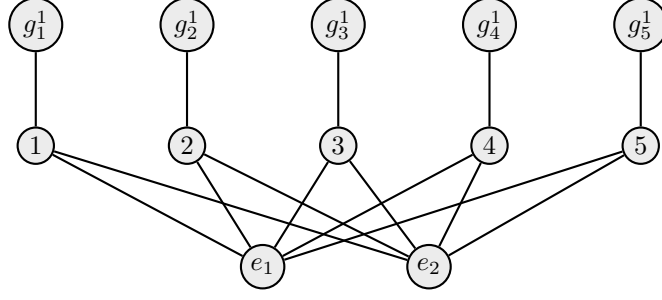
Theorem 23 (Part 1). *The problem of maximizing social welfare under s -step function externalities in a p -neighborhood is APX-hard (even for $s = p$), that is there is no polynomial-time $1 - \frac{1}{e} + \epsilon$ -approximation algorithm (unless $\text{P} = \text{NP}$).*

Proof. The proof is by an approximation ratio preserving reduction from the MAX COVERAGE problem defined below.

► **Definition 36.** The input to the MAX COVERAGE problem is a set of domain elements $D = \{1, 2, \dots, n\}$, together with non-negative integer weights $w_1, \dots, w_n$, a collection $\mathcal{S} = \{E_1, \dots, E_m\}$ of subsets of D and a positive integer k . The goal is to find a collection $\mathcal{S}' \subseteq \mathcal{S}$ of cardinality k maximizing $\sum_{j \in \bigcup_{E \in \mathcal{S}'} E} w_j$.

Now consider the following reduction from instance $(D, \mathcal{S}, k)$ of the MAX COVERAGE problem to our problem. We construct a bipartite graph G with node set V and an item set I with $I = \mathcal{S}$. We build G as follows: For each element $j \in D$ we create one node j in V and $s - 1$ nodes g_j^t for $1 \leq t \leq s - 1$. Additionally V contains k nodes e_l with $1 \leq l \leq k$. There is an edge in G from each node j to every node g_j^t for all $1 \leq t \leq s - 1$. Additionally there is an edge from each node j to every node e_l for $1 \leq l \leq k$. Finally the intrinsic values of an agent $j \in D$ for getting item $E \in \mathcal{S}$ are given by

$$\lambda_{E,j} = \begin{cases} w_j & \text{if } j \in E \\ 0 & \text{otherwise.} \end{cases}$$



■ **Figure 3** An illustration of the reduction for 2-step functions and $(D, \mathcal{S}, k) = (\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}, 2)$. Notice that the graph does not depend on $\mathcal{S}$.

while the intrinsic values of an agent $j \notin D$ are given by $\lambda_{E,j} = 0$ for all E . The valuations of the agents for a given assignment are then given by $u_j(\mathcal{A}) = \lambda_{\mathcal{A}(j),j} \cdot \text{ext}_{\mathcal{A}(j),j}(|S_{\mathcal{A}(j),j}^p(\mathcal{A})|)$, where $\text{ext}_{i,j}(t)$ are s -step function externalities.

► **Lemma 37.** *The above reduction maps each instance of the MAX COVERAGE problem to a network of agents with s -step function externalities in p -neighborhoods such that the value of the optimal selection for the MAX COVERAGE problem is equal to the social welfare of the optimal item assignment.*

Proof. We will show that (a) given a solution for the MAX COVERAGE instance with value W we can construct an assignment for the s -step function externalities in p -neighborhoods with social welfare at least W and (b) given an assignment for the s -step function externalities in p -neighborhoods with social welfare W we can construct a solution for the MAX COVERAGE instance with value at least W . The lemma then follows.

(a) Consider a feasible solution $\mathcal{S}'$ for the MAX COVERAGE instance with value $W = \sum_{j \in \bigcup_{E \in \mathcal{S}'} E} w_j$. W.l.o.g. we can assume $\mathcal{S}' = \{E_1, \dots, E_k\}$. One can construct an item assignment to the nodes as follows: For each $1 \leq l \leq k$ assign item E_l to node e_l and for each $j \in \bigcup_{E \in \mathcal{S}'} E$ pick one $E_l \in \mathcal{S}'$ such that $j \in E$ and assign E_l to j and g_j^t for $1 \leq t \leq s-1$. Finally, assign arbitrary items to the remaining nodes. It remains to show that the social welfare of this assignment is at least W . To this end consider an arbitrary $j \in \bigcup_{E \in \mathcal{S}'} E$. For this j the corresponding node has utility w_j as there are at least s neighbors in his p -neighborhood having the same item. This is because there is a node e_l such that (i) $j \in E_l$ and (ii) e_l, j , and all g_j^t for $1 \leq t \leq s-1$ get the same item E_l . By construction of the graph j is connected to all these s nodes by an edge. As this holds for each $j \in \bigcup_{E \in \mathcal{S}'} E$ the social welfare of the given assignment is at least W .

(b) Given an item assignment $\mathcal{A}$ one can construct a feasible solution for the MAX COVERAGE instance as follows. Consider the nodes $e_1, \dots, e_k$ and the collection $\mathcal{S}'$ of assigned items $\{E_{e_1}, \dots, E_{e_k}\}$. Now if we consider the items E_{e_j} again as subsets of D we have that $\mathcal{S}'$ is a feasible solution for the MAX COVERAGE instance (as $|\mathcal{S}'| \leq k$). Now let D^* be the set of nodes j in the graph with $j \in D$ such that j contributes a positive amount to the social welfare in $\mathcal{A}$. Each node $j \in D^*$ contributes exactly w_j to the social welfare. For each $j \in D^*$ there is at least one node e_l with the same item E_l s.t. $j \in E_l$, as j has only $s-1$ neighbors that are not e_l nodes and none of these neighbors has further edges. By construction $E_l \in \mathcal{S}'$ and thus j is covered by $\mathcal{S}'$ and contributes w_j to the value of $\mathcal{S}'$. As this holds for each $j \in D^*$ the value achieved by the feasible solution $\mathcal{S}'$ is at least as high as the social welfare of the item assignment $\mathcal{A}$. ◀

So, by the above lemma, the presented reduction is approximation ratio preserving and, as it can be also performed in polynomial time, Theorem 23 follows from the corresponding result for MAX COVERAGE [9, 23]. $\blacktriangleleft$

Notice that the above reduction introduces an arbitrary large number m of items. So the critical reader might ask whether the problem admits a PTAS if one bounds the number of items. The answer to this question is no as we can show that the problem is still APX-hard if we just consider two items, however for a smaller constant. So Part 2 of Theorem 23 is by the following theorem.

► **Theorem 38.** *Maximizing social welfare with s -step function externalities in a p -neighborhood is APX-hard even for the case with 2 items, i.e., it is NP-hard to approximate better than a factor of $\frac{23}{24}$.*

For showing APX-hardness we make use of a variant of the MAX-SAT problem, where each clause contains either only positive literals or only negative literals. We first have to show that this variant is still APX-hard, using a reduction from MAX3-SAT which is well-known to be APX-hard.

► **Proposition 39.** *Monotone MAX-SAT is APX-hard, i.e., it is NP-hard to approximate it better than a factor of $\frac{23}{24}$.*

Proof. We show the assertion by a reduction from MAX3-SAT for which it is well-known that approximations better than $7/8$ are NP-hard [19]. To this end we consider the following reduction from [13] which shows the decision version of monotone SAT to be NP-hard.

Assume we are given an instance of MAX3-SAT; i.e., a CNF formula $\varphi = \bigwedge_{\gamma \in \Gamma} \bigvee_{l \in \gamma} l$ with Γ being a collection of clauses, where a clause γ is a set of literals of X .

For each $\gamma \in \Gamma$ let γ_p be the set of positive literals in γ and let γ_n be the set of negative literals in γ . We now construct an instance $\psi = \bigwedge_{\gamma \in \hat{\Gamma}} \gamma$ of Monotone MAX-SAT as follows. For each $\gamma \in \Gamma$ we have $\gamma_p \cup \{z_\gamma\} \in \hat{\Gamma}$ and $\gamma_n \cup \{\neg z_\gamma\} \in \hat{\Gamma}$, where z_γ is a new variable. Clearly each of these clauses contains either only positive literals or only negative literals. Further let m be the number of clauses in φ and M the maximal number of simultaneously satisfied clauses of φ .

Claim: The maximal number of simultaneously satisfied clauses in ψ is $M + m$.

To show the claim let us first assume we have a truth assignment τ satisfying M clauses of φ . We can extend τ to a truth assignment τ' for the Monotone MAX-SAT as follows. If τ satisfies a positive literal of clause γ then set $\tau'(z_\gamma) = \text{false}$ otherwise to $\tau'(z_\gamma) = \text{true}$. Now we have that if τ satisfies clause γ then both corresponding clauses in ψ are satisfied by τ' and if γ is not satisfied by τ then only the positive clause $\gamma_p \cup \{z_\gamma\}$ in ψ is satisfied by τ' . Thus we have that τ' satisfies $M + m$ clauses of ψ .

Now let us assume we have an assignment τ' for ψ satisfying M' clauses. Now consider the assignment τ for variables X which is given by $\tau(x) = \tau'(x)$ for all $x \in X$, i.e., τ is the projection of τ' on X . Consider a clause $\gamma \in \Gamma$ where both corresponding clauses in ψ are true. At most one of these clauses can be satisfied by the variable z_γ and thus γ must be satisfied by τ . Finally, observe that for at least $M' - m$ many clauses of φ both corresponding clauses in ψ are satisfied and thus τ satisfies at least $M' - m$ clauses of φ , which concludes the proof of the claim.

We next show that if we could $(\frac{23}{24} + \epsilon)$ -approximate Monotone MAX-SAT then we could also $(7/8 + \epsilon)$ -approximate MAX3-SAT. Given a MAX3-SAT instance φ and ψ constructed from φ as above, let M be the maximal number of simultaneously satisfied clauses of φ , and let

M' be the maximal number of simultaneously satisfied clauses of ψ . Towards a contradiction assume a $(\frac{23}{24} + \epsilon)$ -approximation algorithm for Monotone MAX-SAT and consider a $(\frac{23}{24} + \epsilon)$ approximation A' that satisfies a' many clauses of ψ . Then, by the above observation, we can construct in polynomial time an assignment for φ that satisfies $a' - m$ many clauses.

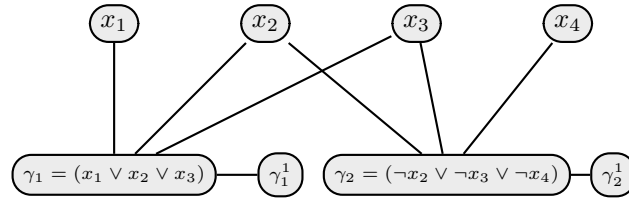
$$\begin{aligned} a' - m &\geq (\frac{23}{24} + \epsilon)M' - m = (\frac{23}{24} + \epsilon)(M + m) - m \\ &\geq (\frac{23}{24} + \epsilon)M - (\frac{1}{24} - \epsilon)m \\ &\geq (\frac{23}{24} + \epsilon)M - (\frac{1}{24} - \epsilon)2M = (\frac{7}{8} + 3\epsilon) \cdot M \end{aligned}$$

For the last inequality we used the fact that in each CNF-formula at least a half of the clauses can be satisfied, i.e., $m \leq 2M$. By the above, any $(\frac{23}{24} + \epsilon)$ -approximation algorithm for Monotone MAX-SAT results in a $(\frac{7}{8} + 3\epsilon)$ -approximation of the MAX3-SAT instance, a contradiction. $\blacktriangleleft$

Proof of Theorem 38. We show hardness by the following reduction from monotone MAX3-SAT. Assume we are given an instance of monotone MAX3-SAT; i.e., a CNF formula $\varphi = \bigwedge_{\gamma \in \Gamma \cup \bar{\Gamma}} \gamma$ with Γ being a set of clauses over variables X containing only positive literals and $\bar{\Gamma}$ being a set of clauses over variables X containing only negative literals. We construct the following network. The agents are given by $V = X \cup \Gamma \cup \bar{\Gamma} \cup \bigcup_{i=1}^{s-1} \Gamma^i \cup \bigcup_{i=1}^{s-1} \bar{\Gamma}^i$, where Γ^i and $\bar{\Gamma}^i$ are copies of Γ and $\bar{\Gamma}$, and there are only two items T and F . The (undirected) edges of G are given by (i) (γ, γ^i) for each $\gamma \in \Gamma \cup \bar{\Gamma}$ and copies $\gamma^i \in \Gamma^i \cup \bar{\Gamma}^i$, and (ii) (x, γ) for each γ and variable x with $x \in \gamma$ or $\neg x \in \gamma$. Next consider the intrinsic values $\lambda_{i,j}$ in the valuation functions. For each $\gamma \in \Gamma$ set $\lambda_{T,\gamma} = 1$, for each $\gamma \in \bar{\Gamma}$ set $\lambda_{F,\gamma} = 1$, and $\lambda_{i,j} = 0$ for all other combinations of $i \in \{T, F\}$ and $j \in V$. The valuations of the agents for a given assignment are then given by $u_j(\mathcal{A}) = \lambda_{\mathcal{A}(j),j} \cdot \text{ext}_{\mathcal{A}(j),j}(|S_{\mathcal{A}(j),j}^p(\mathcal{A})|)$, where $\text{ext}_{i,j}(\mathcal{A})$ are s-step function externalities.

Next we show that the maximal number of simultaneously satisfied clauses of φ is exactly the maximal social welfare one can obtain in the network.

1. Consider a truth assignment τ for φ and construct the following item assignment. (i) Agent $x \in X$ gets item T if $\tau(x) = \text{true}$ and $x \in X$ gets item F if $\tau(x) = \text{false}$. (ii) For $\gamma \in \Gamma$ agent γ and γ 's copies γ^i get item T , and for $\gamma \in \bar{\Gamma}$ agents γ and γ 's copies γ^i get item F . Consider a clause $\gamma \in \Gamma$ satisfied by τ via literal x . Then the valuation of agent γ is one as he has s neighbors, $\gamma^1, \dots, \gamma^{s-1}$ and x , that also get item T . Now consider a clause $\gamma \in \bar{\Gamma}$ satisfied by τ via literal $\neg x$. Then the valuation of agent γ is one as he has s neighbors, $\gamma^1, \dots, \gamma^{s-1}$ and x , that also get item F . Thus for each satisfied clause we achieve a valuation of one.
2. Consider an item assignment and the truth assignment τ defined by $\tau(x) = \text{true}$ iff agent x gets item T . Notice that we can only achieve positive valuation from agents $\gamma \in \Gamma \cup \bar{\Gamma}$. First consider a agent $\gamma \in \Gamma$ that contributes to the social welfare. Then, as γ has only $s - 1$ neighbors not in X and those neighbors have no neighbors except γ , there has to exist an $x \in \gamma$ such that agent x gets item T . By definition of τ we then have that τ satisfies the clause γ . Second consider an agent $\gamma \in \bar{\Gamma}$ that contributes to the social welfare. Then there has to exist an x with $\neg x \in \gamma$ such that agent x gets item F . Again by the definition of τ we then have that τ satisfies the clause γ . Thus, we have that for each agent that contributes to the social welfare we satisfy one clause and as each of these agent contributes exactly 1 the claim follows.



■ **Figure 4** An illustration of the reduction for $(x_1 \vee x_2 \vee x_3) \wedge (\neg x_2 \vee \neg x_3 \vee \neg x_4)$ and $s = 2$.

Now as we have a one-to-one correspondence between the number of satisfied clauses and the social welfare, not only NP-hardness but also hardness of approximation follows from the corresponding result for monotone MAX3-SAT. ◀