# Ad Exchange: Envy-Free Auctions with Mediators*

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**Abstract.** Ad exchanges are an emerging platform for trading advertisement slots on the web with billions of dollars revenue per year. Every time a user visits a web page, the publisher of that web page can ask an ad exchange to auction off the ad slots on this page to determine which advertisements are shown at which price. Due to the high volume of traffic, ad networks typically act as mediators for individual advertisers at ad exchanges. If multiple advertisers in an ad network are interested in the ad slots of the same auction, the ad network might use a "local" auction to resell the obtained ad slots among its advertisers.

In this work we want to deepen the theoretical understanding of these new markets by analyzing them from the viewpoint of combinatorial auctions. Prior work studied mostly single-item auctions, while we allow the advertisers to express richer preferences over multiple items. We develop a game-theoretic model for the entanglement of the *central* auction at the ad exchange with the *local* auctions at the ad networks. We consider the incentives of all three involved parties and suggest a *three-party competitive equilibrium*, an extension of the Walrasian equilibrium that ensures envy-freeness for all participants. We show the existence of a three-party competitive equilibrium and a polynomial-time algorithm to find one for gross-substitute bidder valuations.

**Keywords:** ad-exchange, combinatorial auctions, gross substitute, Walrasian equilibrium, three-party equilibrium, auctions with mediators

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* This work was funded by the Vienna Science and Technology Fund (WWTF) through project ICT10-002. Additionally the research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement no. 340506. A full version of the paper is available at https://eprints.cs.univie.ac.at/4456/.

** Work done while at the University of Vienna.

## 1 Introduction

As advertising on the web becomes more mature, *ad exchanges* (AdX) play a growing role as a platform for selling advertisement slots from publishers to advertisers. Following the Yahoo! acquisition of Right Media in 2007, all major web companies, such as Google, Facebook, and Amazon, have created or acquired their own ad exchanges. Other major ad exchanges are provided by the Rubicon Project, OpenX, and AppNexus. In 2012 the total revenue at ad exchanges was estimated to be around two billion dollars [6]. Every time a user visits a web page, the publisher of that web page can ask an ad exchange to auction off the ad slots on this page. Thus, the goods traded at an ad exchange are *ad impressions*. This process is also known as *real-time bidding* (RTB). A web page might contain multiple ad slots, which are currently modeled to be sold separately in individual auctions. Individual advertisers typically do not directly participate in these auctions but entrust some ad network to bid on their behalf. When a publisher sends an ad impression to an exchange, the exchange usually contacts several ad networks and runs a (variant of a) second-price auction [17] between them, potentially with a reserve price under which the impression is not sold. An ad network (e.g. Google's Display Network [9]) might then run a second, "local" auction to determine the allocation of the ad slot among its advertisers. We study this interaction of a *central auction* at the exchange and *local auctions* at the ad networks.[3]

We develop a game-theoretic model that considers the incentives of the following three parties: (1) the ad exchange, (2) the ad networks, and (3) the advertisers. As the ad exchange usually charges a fixed percentage of the revenue and hands the rest to the publishers, the ad exchange and the publishers have the same objective and can be modeled as one entity. We then study equilibrium concepts of this new model of a three-party exchange. Our model is described as an ad exchange, but it may also model other scenarios with mediators that act between bidders and sellers, as noted already by Feldman et al. [7]. The main differences between our model and earlier models (discussed in detail at the end of this section) are the following: (a) We consider the incentives of all three parties *simultaneously*. (b) While most approaches in prior work use Bayesian assumptions, we apply *worst-case analysis*. (c) We allow auctions with *multiple heterogeneous items*, namely combinatorial auctions, in contrast to the single-item auctions studied so far. Multiple items arise naturally when selling ad slots on a per-impression basis, since there are usually multiple advertisement slots on a web page.

To motivate the incentives of ad networks and exchanges, we compare next their short and long-term revenue considerations, following Mansour et al. [17] and Muthukrishnan [19]. Ad exchanges and ad networks generate revenue as follows: (1) An ad exchange usually receives some percentage of the price paid by the winner(s) of the central auction. (2) An ad network can charge a higher

[3] In this work an auction is an algorithm to determine prices of items and their allocation to bidders.

price to its advertisers than it paid to the exchange or it can be paid via direct contracts with its advertisers. Thus both the ad exchange and the ad networks (might) profit from higher prices in their auctions. However, they also have a motivation not to charge too high prices as (a) the advertisers could stick to alternative advertising channels such as long-term contracts with publishers, and (b) there is a significant competition between the various ad exchanges and ad networks, as advertisers can easily switch to a competitor. Thus, lower prices (might) increase advertiser participation and, hence, the long-term revenue of ad exchanges and ad networks. We only consider a single auction (of multiple items) and leave it as an open question to study changes over time. We still take the long-term considerations outlined above into account by assuming that the ad exchange aligns its strategic behavior with its long-term revenue considerations and only desires for each central auction to sell *all* items.[4] In our model the incentive of an ad network to participate in the exchange comes from the opportunity to purchase some items at a low price and then resell them at a higher price. However, due to long-term considerations, our model additionally requires the ad networks to "satisfy their advertisers" by faithfully representing the advertisers' preferences towards the exchange, while still allowing the ad networks to extract revenue from the competition between the advertisers in their network.[5] An example for this kind of restriction for an ad network is Google's Display Network [9] that guarantees its advertisers that each ad impression is sold via a second-price auction, independent of whether an ad exchange is involved in the transaction or not [17].

To model a *stable* outcome in a three-party exchange, we use the equilibrium concept of *envy-freeness* for all three types of participants. A participant is envy-free if he receives his most preferred set of items under the current prices. Envy-freeness for all participants is a natural notion to express stability in a market, as it implies that no coalition of participants would strictly profit from deviating from the current allocation and prices (assuming truthfully reported preferences). Thus an envy-free equilibrium supports stability in the market prices, which in turn facilitates, for example, revenue prediction for prospective participants and hence might increase participation and long-term revenue. For only two parties, i.e., sellers and buyers, where the sellers have no intrinsic value for the items they sell, envy-freeness for all participants is equal to a *competitive* or *Walrasian* equilibrium [25], a well established notion in economics to characterize an equilibrium in a market where demand equals supply. We provide a generalization of this equilibrium concept to three parties.

*Our Contribution.* We introduce the following model for ad exchanges. A *central seller* wants to sell $k$ *items*. There are $m$ *mediators* $\mathcal{M}_i$, each with her own $n_i$

[4] Our model and results can be adapted to include reserve prices under which the ad exchange is not willing to sell an item.

[5] We implicitly assume that the central auction prices are accessible to the advertisers such that they can verify whether an ad network represented their preferences correctly. Informally, we suggest that if one ad network "satisfies its advertisers" then, over time, all ad networks have to follow this behavior to keep their advertisers.

*bidders*. Each bidder has a valuation function over the items. In the ad exchange setting, the central seller is the ad exchange, the items are the ad slots shown to a visitor of a web page, the mediators are the ad networks, and the bidders are the advertisers. A bidder does not have any direct "connection" to the central seller. Instead, all communication is done through the mediators. A mechanism for allocating the items to the bidders is composed of a *central auction* with mediators acting as bidders, and then *local auctions*, one per mediator, in which every mediator allocates the set of items she bought in the central auction; that is, an auction where the bidders of that mediator are the only participating bidders and the items that the mediator received in the central auction are the sole items. The prices of the items obtained in the central auction provide a lower bound for the prices in the local auctions, i.e., they act as reserve prices in the local auctions. We assume that the central seller and the bidders have quasi-linear utilities, i.e., utility functions that are linear in the price, and that their incentive is to maximize their utility. For the central seller this means that his utility from selling a set of slots is just the sum of prices of the items in the set. The utility of a bidder on receiving a set of items $S$ is his value for $S$ minus the sum of the prices of the items in $S$.

The incentive of a mediator, however, is not so straightforward and needs to be defined carefully. In our model, to "satisfy" her bidders, each mediator guarantees her bidders that the outcome of the local auction will be *minimal envy free*, that is, for the final local price vector, the item set that is allocated to any bidder is one of his most desirable sets over *all* possible item sets (even sets that contain items that were not allocated to his mediator, i.e., each bidder is not only *locally*, but *globally envy-free*) and there is no (item-wise) smaller price vector that fulfills this requirement. We assume that each mediator wants to maximize her revenue[6] and define the revenue of a mediator for a set of items $S$ as the difference between her earnings when selling $S$ with this restriction and the price she has to pay for $S$ at the central auction.

For this model we define a new equilibrium concept, namely the *three-party competitive equilibrium*. At this equilibrium all three types of participants are envy-free. Envy-free solutions for the bidders always exist, as one can set the prices of all items high enough so that no bidder will demand any item. Additionally, we require that there is no envy for the central seller, meaning that all items are sold. If there were no mediators, then a two-party envy-free solution would be exactly a *Walrasian equilibrium*, which for certain scenarios can be guaranteed [15]. However, with mediators it is not a-priori clear that a three-party competitive equilibrium exists as, additionally, the mediators have to be envy-free. We show that for our definition of a mediator's revenue (a) the above requirements are fulfilled and (b) a three-party competitive equilibrium exists whenever a Walrasian equilibrium for the central auction exists or whenever a two-party equilibrium exists for the bidders and the central seller without mediators. Interestingly, we show that for gross-substitute bidder valuations the incentives of this kind of mediator can be represented with an OR-valuation

[6] For the purpose of this paper, the terms revenue and utility are interchangeable.

over the valuations of her bidders. This then leads to the following result: *For gross-substitute bidder valuations a three-party competitive equilibrium can be computed in polynomial time.* In particular, we will show how to compute the three-party competitive equilibrium with minimum prices.

*Related Work.* The theoretical research on ad exchanges was initialized by a survey of Muthukrishnan [19] that lists several interesting research directions. Our approach specifically addresses his 9th problem, namely to enable the advertisers to express more complex preferences that arise when multiple advertisement slots are auctioned off at once as well as to design suitable auctions for the exchange and the ad networks to determine allocation and prices given these preferences.

The most closely related work with respect to the model of the ad exchange is Feldman et al. [7]. It is similar to our work in two aspects: (1) The mediator bids on behalf of her bidders in a central auction and the demand of the mediator as well as the tentative allocation and prices for reselling to her bidders are determined via a local auction. (2) The revenue of the mediator is the price she can obtain from reselling minus the price she paid in the central auction. The main differences are: (a) Only one item is auctioned at a time and thus the mediator can determine her valuation with a single local auction. (b) Their work does not consider the incentives of the bidders, only of the mediators and the central seller. (c) A Bayesian setting is used where the mediators and the exchange know the probability distributions of the bidders' valuations. Based on this information, the mediators and the exchange choose reserve prices for their second-price auctions to maximize their revenue. The work characterizes the equilibrium strategies for the selection of the reserve prices.

Mansour et al. [17] (mainly) describe the auction at the DoubleClick exchange. Similar to our work advertisers use ad networks as mediators for the central auction. They observe that if mediators that participate in a single-item, second-price central auction are only allowed to submit a single bid, then it is not possible for the central auction to correctly implement a second-price auction over *all* bidders as the bidders with the highest and the second highest value might use the same mediator. Thus they introduce the *Optional Second Price* auction, where every mediator is allowed to optionally submit the second highest bid with the highest bid. In such an auction each mediator can guarantee to her bidders that if one of them is allocated the item, then he pays the (global) second-price for it. For the single-item setting, the bidders in their auction and in our auction pay the same price. If the mediator of the winning bidder did *not* specify an optional second price, then her revenue will equal the revenue of our mediator. If she did, her revenue will be zero and the central seller will receive the gain between the prices in the local and the central auction.

Stavrogiannis et al. [23] consider a game between bidders and mediators, where the bidders can select mediators (based on Bayesian assumptions of each other's valuations) and the mediators can set the reserve prices in the second-price local auction. The work presents mixed Nash equilibrium strategies for the bidders to select their mediator. In [24] the same authors compare different singel-item local auctions with respect to the achieved social welfare and the revenue of the mediators and the exchange.

Balseiro et al. (2013) introduced a setting that does *not* include mediators [1]. Instead, they see the ad exchange as a game between publishers, who select parameters such as reserve prices for second-price auctions, and advertisers, whose budget constraints link different auctions over time. They introduced a new equilibrium concept for this game and used this to analyze the impact of auction design questions such as the selection of a reserve price. Balseiro et al. (2014) studied a publisher's trade-off between using an ad exchange versus fulfilling long-term contracts with advertisers [2].

Equilibria in trading networks (such as ad exchanges) are also addressed in the "matching with contracts" literature. Hatfield and Milgrom [14] presented a new model where instead of bidders and items there are *agents* and *trades* between pairs of agents. The potential trades are modeled as edges in a graph where the agents are represented by the nodes. Agent valuations are then defined over the potential trades and assumed to be monotone substitute. They proved the existence of an (envy-free) equilibrium when the agent-trades graph is bipartite. Later this was improved to directed acyclic graphs by Ostrovsky [21] and to arbitrary graphs by Hatfield et al. [13]. They did not show (polynomial-time) algorithms to reach equilibria. Our model can be reduced to this model, hence a three-party equilibrium exists when all bidders are monotone gross substitute. The result of this reduction (not stated here) is not polynomial in the number of bidders and items.

*Outline.* After the preliminaries in Sect. 2, we formally define our model for ad exchanges in Sect. 3. In particular we define the three-party competitive equilibrium and the mediator model of our choice. We give two simple existence proofs for the three-party equilibrium; one under the condition that a two-party equilibrium between the bidders and the central seller exists, and one for the case that a two-party equilibrium between the central seller and the mediators exists. In Sect. 4 we focus on gross-substitute bidders and show that in this case our mediator definition is equivalent to a valuation-based OR-player definition and how this implies a polynomial-time algorithm to compute a three-party competitive equilibrium. Finally we discuss our results in Sect. 5. All omitted proofs are given in the appendix.

## 2 Preliminaries

Let $\Omega$ denote a set of $k$ items. A *price vector* is an assignment of a non-negative price to every element of $\Omega$. For a price vector $p = (p_1, ..., p_k)$ and a set $S \subseteq \Omega$ we use $p(S) = \sum_{j \in S} p_j$. For any two price vectors $p, r$ an inequality such as $p \geq r$ as well as the operations $\min(p, r)$ and $\max(p, r)$ are meant item-wise.

We denote with $\langle \Omega_b \rangle = \langle \Omega_b \rangle_{b \in \mathcal{B}}$ an *allocation* of the items in $\Omega$ such that for all bidders $b \in \mathcal{B}$ the set of items allocated to $b$ is given by $\Omega_b$ and we have $\Omega_b \subseteq \Omega$ and $\Omega_b \cap \Omega_{b'} = \emptyset$ for $b' \neq b$, $b' \in \mathcal{B}$. Note that some items might not be allocated to any bidder.

A *valuation* function $v_b$ of a bidder $b$ is a function from $2^{\Omega}$ to $\mathbb{R}$, where $2^{\Omega}$ denotes the set of all subsets of $\Omega$. We assume throughout the paper $v_b(\emptyset) = 0$. Unless specified otherwise, for this work we assume *monotone* valuations, that is, for $S \subseteq T$ we have $v_b(S) \leq v_b(T)$. This assumption is made for ease of presentation. We use $\{v_b\}$ to denote a collection of valuation functions. The *utility* of a bidder $b$ from a set $S \subseteq \Omega$ at prices $p \geq 0$ is defined as $u_{b,p}(S) = v_b(S) - p(S)$. Such utility functions are often called *quasi-linear*, i.e., linear in the price. The *demand* $D_b(p)$ of a bidder $b$ for prices $p \geq 0$ is the set of subsets of items $S \subseteq \Omega$ that maximize the bidder's utility at prices $p$. We call a set in the demand a *demand representative*. Throughout the paper we omit subscripts if they are clear from the context.

**Definition 1 (Envy free).** *An allocation $\langle \Omega_b \rangle$ of items $\Omega$ to bidders $\mathcal{B}$ is envy free (on $\Omega$) for some prices $p$ if for all bidders $b \in \mathcal{B}$, $\Omega_b \in D_b(p)$. We say that prices $p$ are envy free (on $\Omega$) if there exists an envy-free allocation (on $\Omega$) for these prices.*

There exist envy-free prices for any valuation functions of the bidders, e.g., set all prices to $\max_{b,S} v_b(S)$. For these prices the allocation which does not allocate any item is envy free. Thus also minimal envy-free prices always exist, but are in general not unique.

**Definition 2 (Walrasian equilibrium (WE)).** *A Walrasian equilibrium (on $\Omega$) is an envy-free allocation $\langle \Omega_b \rangle$ (on $\Omega$) with prices $p$ such that all prices are non-negative and the price of unallocated items is zero. We call the allocation $\langle \Omega_b \rangle$ a Walrasian allocation (on $\Omega$) and the prices $p$ Walrasian prices (on $\Omega$).*

We assume that the central seller has a value of zero for every subset of the items; thus (with quasi-linear utility functions) selling all items makes the seller envy free. In this case a Walrasian equilibrium can be seen as an *envy-free two-party equilibrium*, i.e., envy free for the buyers and the seller. Note that for a Walrasian price vector there might exist multiple envy-free allocations.

### 2.1 Valuation Classes

A *unit demand* valuation assigns a value to every item and defines the value of a set as the *maximum* value of an item in it. An *additive* valuation also assigns a value to every item but defines the value of a set as the *sum* of the values of the items in the set. Non-negative unit demand and non-negative additive valuations both have the gross-substitute property (defined below) and are by definition monotone.

**Definition 3 (Gross substitute (gs)).** *A valuation function is* gross substitute *if for every two price vectors $p^{(2)} \geq p^{(1)} \geq 0$ and every set $D^{(1)} \in D(p^{(1)})$, there exists a set $D^{(2)} \in D(p^{(2)})$ with $j \in D^{(2)}$ for every $j \in D^{(1)}$ with $p_j^{(1)} = p_j^{(2)}$.*

For *gross-substitute* valuations of the bidders a Walrasian equilibrium is guaranteed to exist in a two-sided market [15] and can be computed in polynomial time [20, 22]. Further, gross substitute is the maximal valuation class containing the unit demand class for which the former holds [10]. Several equivalent definitions are known for this class [10, 22]. We will further use that for gross-substitute valuations the Walrasian prices form a complete lattice [10].

We define next an OR-valuation. Lehmann et al. [16] showed that the OR of gross-substitute valuations is gross substitute.

**Definition 4 (or-player).** *The* OR *of two valuations* $v$ *and* $w$ *is defined as* $(v \text{ OR } w)(S) = \max_{R,T \subseteq S, R \cap T = \emptyset}(v(R) + w(T))$. *Given a set of valuations* $\{v_b\}$ *for bidders* $b \in \mathcal{B}$ *we say that the* OR*-player is a player with valuation* $v_{\text{OR}}(S) = \max_{\langle S_b \rangle} \sum_{b \in \mathcal{B}} v_b(S_b)$.

## 3 Model and Equilibrium

There are $k$ items to be allocated to $m$ mediators. Each mediator $\mathcal{M}_i$ represents a set $\mathcal{B}_i$ of bidders, where $|\mathcal{B}_i| = n_i$. Each bidder is connected to a unique mediator. Each bidder has a valuation function over the set of items and a quasi-linear utility function. A *central auction* is an auction run on all items with mediators as bidders. After an allocation $\langle \Omega_i \rangle$ and prices $r$ at the central auction are set, another $m$ *local auctions* are conducted, one by each mediator. In the local auction for mediator $\mathcal{M}_i$ the items $\Omega_i$ that were allocated to her in the central auction are the sole items and the bidders $\mathcal{B}_i$ are the sole bidders. A solution is an assignment of central-auction and local-auction prices to items and an allocation of items to bidders and hence, by uniqueness, also to mediators. We define next a three-party equilibrium based on envy-freeness.

**Definition 5 (Equilibrium).** *A* three-party competitive equilibrium *is an allocation of items to bidders and a set of* $m+1$ *price vectors* $r, p^1, p^2, \ldots, p^m$ *such that the following requirements hold. For* $1 \leq i \leq m$

1. *every mediator*[7] $\mathcal{M}_i$ *is allocated a set* $\Omega_i$ *in her demand at price* $r$,
2. *every item* $j$ *with non-zero price* $r$ *is allocated to a mediator,*
3. *the price* $p^i$ *coincides with* $r$ *for all items not in* $\Omega_i$,
4. *and every bidder* $b \in \mathcal{B}_i$ *is allocated a subset of* $\Omega_i$ *that is in his demand at price* $p^i$.

In other words, the allocation to the bidders in $\mathcal{B}_i$ with prices $p^i$ must be envy-free for the bidders, the allocation to the mediators with prices $r$ must be envy free for the mediators and for the central seller, i.e., must be a Walrasian equilibrium; and the prices $p^i$ must be equal to the prices $r$ for all items not assigned to mediator $\mathcal{M}_i$.

Note that the allocation of the items to the mediators and prices $r$ are the outcome of a *central* auction run by the central seller, while the allocation to the

[7] Regardless of any demand definition

bidders in $\mathcal{B}_i$ and prices $p^i$ correspond to the outcome of a *local* auction run by mediator $\mathcal{M}_i$. These auctions are connected by the demands of the mediators and Requirement 3.

We next present our mediator model. The definition of an Envy-Free Mediator, or EF-mediator for short, reflects the following idea: To determine her revenue for a set of items $S$ at central auction prices $r$, the mediator simulates the local auction she would run if she would obtain the set $S$ at prices $r$. Given the outcome of this "virtual auction", she can compute her potential revenue for $S$ and $r$ as the difference between the virtual auction prices of the items sold in the virtual auction and the central auction prices for the items in $S$. However, as motivated in the introduction, the mediator is required to represent the preferences of her bidders and therefore not every set $S$ is "allowed" for the mediator, that is, for some sets the revenue of the mediator is set to $-1$. The sets that maximize the revenue are then in the demand of the mediator at central auction prices $r$. To make the revenue of a mediator well-defined and to follow our motivation that a mediator should satisfy her bidders, the virtual auctions specifically compute minimal envy-free price vectors.

**Definition 6 (Envy-Free Mediator).** *An* EF*-mediator $\mathcal{M}_i$ determines her demand for a price vector $r \geq 0$ as follows. For each subset of items $S \subseteq \Omega$ she runs a virtual auction with items $S$, her bidders $\mathcal{B}_i$, and reserve prices $r$. We assume that the virtual auction computes minimal envy-free prices $p^S \geq r$ and a corresponding envy-free allocation $\langle S_b \rangle$. We extend the prices $p^S$ to all items in $\Omega$ by setting $p^S_j = r_j$ for $j \in \Omega \setminus S$, and define the revenue $R_{i,r}(S)$ of the mediator for a set $S$ as follows. If the allocation $\langle S_b \rangle$ is envy free for the bidders $\mathcal{B}_i$ and prices $p^S$ on $\Omega$, then $R_{i,r}(S) = \sum_{b \in \mathcal{B}_i} p^S(S_b) - r(S)$; otherwise, we set $R_{i,r}(S) = -1$.[8] The demand $D_i(r)$ of $\mathcal{M}_i$ is the set of all sets $S$ that maximize the revenue of the mediator for the reserve prices $r$. The local auction of $\mathcal{M}_i$ for a set $\Omega_i$ allocated to her in the central auction at prices $r$ is equal to her virtual auction for $\Omega_i$ and $r$.*

Note that for a set $S$ with $R_{i,r}(S) = \sum_{b \in \mathcal{B}_i} p^S(S_b) - r(S)$ the revenue of an EF-mediator $\mathcal{M}_i$ is maximal if the envy-free allocation on $S$ is such that $\sum_{b \in \mathcal{B}_i} p^S(S_b)$ is as high as possible. Thus if there are multiple envy-free allocations on $S$ for the prices $p^S$, the mediator chooses one that maximizes $\sum_{b \in \mathcal{B}_i} p^S(S_b)$.

Following the above definition, we say that a price vector is *locally envy free* if it is envy free for the bidders $\mathcal{B}_i$ on the subset $\Omega_i \subseteq \Omega$ assigned to mediator $\mathcal{M}_i$ and *globally envy free* if it is envy free for the bidders $\mathcal{B}_i$ on $\Omega$. Note that if $p^S$ is envy free on $\Omega$, then it is minimal envy free $\geq r$ on $\Omega$ for the bidders $\mathcal{B}_i$.

An interesting property of EF-mediators is that every Walrasian equilibrium in the central auction can be combined with the outcome of the local auctions of EF-mediators to form a three-party competitive equilibrium.

[8] For the results of this paper this could be any negative value including $-\infty$.

**Theorem 1.** *Assume all mediators are* EF*-mediators. Then a Walrasian equilibrium in the central auction with allocation $\langle \Omega_i \rangle$ together with the allocation and prices computed in the local auctions of the mediators $\mathcal{M}_i$ on their sets $\Omega_i$ (not necessarily Walrasian) form a three-party competitive equilibrium.*

Further, with EF-mediators a three-party competitive equilibrium exists whenever a Walrasian eq. exists for the bidders and items without the mediators.

**Theorem 2.** *Assume all mediators are* EF*-mediators and a Walrasian equilibrium exists for the set of bidders and items (without mediators). Then there exists a three-party competitive equilibrium.*

The proof of Theorem 2 only shows the existence of trivial three-party equilibria that basically ignores the presence of mediators. However, three-party equilibria and EF-mediators allow for richer outcomes that permit the mediators to gain revenue from the competition between their bidders while still representing the preferences of their bidders towards the central seller. In the next section we show how to find such an equilibrium provided that the valuations of all bidders are gross substitute. Recall that gross-substitute valuations are the most general valuations that include unit demand valuations for which a Walrasian equilibrium exists [10]; and that efficient algorithms for finding a Walrasian equilibrium are only known for this valuation class.

## 4 An Efficient Algorithm for Gross-substitute Bidders

In this section we will show how to find, in polynomial time, a three-party competitive equilibrium if the valuations of all bidders are gross substitute. The prices the bidders have to pay at equilibrium, and thus the utilities they achieve, will be the same as in a Walrasian equilibrium (between bidders and items) with minimum prices (Appendix E). The price the bidders pay is split between the mediators and the exchange. We show how to compute an equilibrium where this split is best for the mediators and worst for the exchange. In turn the computational load can be split between the mediators and the exchange as well. The algorithm will be based on existing algorithms to compute Walrasian equilibria for gross-substitute bidders.

The classical (two-party) *allocation problem* is the following: We are given $k$ items and $n$ valuation functions and we should find an equilibrium allocation (with or without equilibrium prices) if one exists. Recall that in general a valuation function has a description of size exponential in $k$. Therefore, the input valuation functions can only be accessed via an *oracle*, defined below. An *efficient* algorithm runs in time polynomial in $n$ and $k$ (where the oracle access is assumed to take constant time).

Given an algorithm that computes a Walrasian allocation for gross-substitute bidders, by a result of Gul and Stacchetti [10] minimum Walrasian prices can be computed by solving the allocation problem $k+1$ times. A Walrasian allocation can be combined with any Walrasian prices to form a Walrasian equilibrium [10].

Thus we can assume for gross-substitute valuations that a polynomial-time algorithm for the allocation problem also returns a vector of minimum prices that support the allocation.

Two main oracle definitions that were considered in the literature are the *valuation oracle*, where a query is a set of items $S$ and the oracle replies with the exact value of $S$; and the *demand oracle*, where a query is a price vector $p$ and the oracle replies with a demand representative $D$ [5]. Note that in the literature the answer of a demand oracle is sometimes defined to be all sets in the demand, however this cannot be assumed to be of polynomial size even for gross-substitute valuations.

It is known that a demand oracle is strictly stronger than a valuation oracle, i.e., a valuation query can be simulated by a polynomial number of demand queries but not vice versa. For gross-substitute valuations, however, these two query models are polynomial-time equivalent, see Paes Leme [22]. The two-party allocation problem is efficiently solvable for gross-substitute valuations [20, 22]. For other valuations, efficient algorithms are not known even in the demand query model.

We define the *three-party allocation problem* in the same manner. We are given $k$ items, $n$ valuation functions over the items and $m$ mediators, each associated with a set of unique bidders. We are looking for a three-party equilibrium allocation (and equilibrium prices) if one exists. We will assume that the input valuations are given through a valuation oracle. An efficient algorithm runs in time polynomial in $n$ and $k$ (hence also in $m \leq n$).

The algorithm will be based on the following central result: For gross substitute valuations of the bidders an EF-mediator and an OR-player over the valuations of the same bidders are equivalent with respect to their demand and their allocation of items to bidders. Thus in this case EF-mediators can be considered as if they have a gross-substitute valuation. Note that for general valuations this equivalence does not hold.

**Theorem 3.** *If the valuation functions of a set of bidders $\mathcal{B}_i$ are gross substitute, then the demand of an* EF*-mediator for $\mathcal{B}_i$ is equal to the demand of an* OR*-player for $\mathcal{B}_i$. Moreover, the allocation in a virtual auction of the* EF*-mediator for reserve prices $r$ and a set of items $S$ in the demand is an optimal allocation for the* OR*-player for $S$ and $r$ and vice versa.*

To this end, we will first show for the virtual (and local) auctions that a modified Walrasian equilibrium, the RESERVE-WE(r), exists for gross-substitute valuations with reserve prices. For this we will use yet another reduction to a (standard) Walrasian equilibrium without reserve prices but with an additional additive player[9].

**Definition 7 (Walrasian equilibrium with reserve prices $r$ (reserve-we(r)) [12]).** *A Walrasian equilibrium with reserve prices $r \geq 0$ (on $\Omega$) is*

[9] Such a player was introduced by Paes Leme [22] to find the demand of an OR-player (with a slightly different definition of OR).

*an envy-free allocation* $\langle \Omega_b \rangle$ *(on* $\Omega$*) with prices* $p$ *such that* $p \geq r$*, and the price of every unallocated item is equal to its reserve price, i.e.,* $p_j = r_j$ *for* $j \notin \cup_b \Omega_b$*. We say that* $\langle \Omega_b \rangle$ *is a* RESERVE-WE$(r)$ *allocation (on* $\Omega$*) and* $p$ *are* RESERVE-WE$(r)$ *prices (on* $\Omega$*).*

### 4.1 Properties of Walrasian Equilibria with Reserve Prices

In this section we generalize several results about Walrasian equilibria to Walrasian equilibria with reserve prices. Similar extensions were shown for unit demand valuations in [12].

We first define a suitable linear program. The RESERVE-LP(r) is a linear program obtained from a reformulation of the dual of the LP-relaxation of the welfare maximization integer program after adding reserve prices $r \geq 0$. More details on this reformulation are given in Appendix C.

For an integral solution to the RESERVE-LP(r) we can interpret this reformulation as a solution to a WELFARE-LP with an additional additive player whose value for an item is equal to that item's reserve price. We will use this interpretation to extend known results for Walrasian equilibria to Walrasian equilibria with reserve prices. The results are summarized in Theorem 4 below. We use the following definition.

**Definition 8 (additional additive player).** *Let* $\{v_b\}$ *be a set of valuation functions over* $\Omega$ *for bidders* $b \in \mathcal{B}$*, and let* $r \geq 0$ *be reserve prices for the items in* $\Omega$*. Let* $\{v'_{b'}\}$ *be the set of valuation functions when an additive bidder* $a$ *is added, i.e., for the bidders* $b' \in \mathcal{B}' = \mathcal{B} \cup \{a\}$ *with* $v'_{b'}(S) = v_{b'}(S)$ *for* $b' \neq a$ *and* $v'_a(S) = \sum_{j \in S} r_j$ *for all sets* $S \subseteq \Omega$*. For an allocation* $\langle \Omega_b \rangle_{b \in \mathcal{B}}$ *we define* $\langle \Omega'_{b'} \rangle_{b' \in \mathcal{B}'}$ *with* $\Omega'_{b'} = \Omega_{b'}$ *for* $b' \neq a$ *and* $\Omega'_a = \Omega \setminus \cup_b \Omega_b$*.*

**Theorem 4.** *(a) The allocation* $\langle \Omega_b \rangle$ *and the prices* $p$ *are a* RESERVE-WE$(r)$ *for* $r \geq 0$ *and bidders* $\mathcal{B}$ *if and only if the allocation* $\langle \Omega'_{b'} \rangle$ *and prices* $p'$ *are a* WE *for the bidders* $\mathcal{B}'$*, where we have* $p_j = p'_j$ *for* $j \in \cup_{b \in \mathcal{B}} \Omega_b$ *and* $p_{j'} = r_{j'}$ *for* $j' \in \Omega \setminus \cup_{b \in \mathcal{B}} \Omega_b$ *(a1). The allocation* $\langle \Omega_b \rangle$ *is a* RESERVE-WE$(r)$ *allocation if and only if* $\langle \Omega_b \rangle$ *is an integral solution to the* RESERVE-LP$(r)$ *(a2).*

*(b) If the valuations* $\{v\}$ *are gross substitute, then (b1) there exists a* RESERVE-WE$(r)$ *for* $\{v\}$ *and (b2) the* RESERVE-WE$(r)$ *price vectors form a complete lattice.*

Theorem 4 will be used in the next section to characterize the outcome of the virtual auctions of an EF-mediator. It also provides a polynomial-time algorithm to compute a RESERVE-WE(r) when the bidders in $\mathcal{B}$ have gross-substitute valuations, given a polynomial-time algorithm for a WE for gross-substitute bidders.

### 4.2 The Equivalence of the EF-mediator and the OR-player for Gross-substitute Valuations—Proof Outline

In this section we outline the proof of Theorem 3, that is, the equivalence for gross-substitute bidders between the demand of an EF-mediator $\mathcal{M}_i$ and the demand of an OR-player and, for the sets in the demand, the equivalence of the *allocations* of items to bidders of an OR-player and an EF-mediator in the sense that the allocation implied by the OR-player could be used by the EF-mediator and vice versa. The complete proof is given in Appendix D.

The proof proceeds as follows. We first characterize the demand of an EF-mediator for bidders with gross-substitute valuations. As a first step we show that for such bidders an EF-mediator actually computes a RESERVE-WE(r) with minimum prices in each of her virtual auctions. The minimality of the prices implies that whenever the virtual auction prices for an item set $S$ are globally envy-free, they are also minimum RESERVE-WE(r) prices for the set of all items $\Omega$ and the bidders in $\mathcal{B}_i$. Thus, given reserve prices $r$, all virtual auctions of an EF-mediator result in the same price vector $p$ as long as they are run on a set $S$ with non-negative revenue. With the help of some technical lemmata we then completely characterize the demand of an EF-mediator and show that the mediator does not have to run *multiple* virtual auctions to determine her demand; it suffices to run *one* virtual auction on $\Omega$ where the set of allocated items is a set in the demand of the EF-mediator. Thus for gross-substitute bidders the mediator can efficiently answer demand queries and compute the outcome of her local auction.

Finally we compare the utility function of the OR-player to the optimal value of the RESERVE-LP(r) to observe that they have to be equal (up to an additive constant) for item sets that are in the demand of the OR-player. Combined with the above characterization of the demand of the mediator, we can then relate both demands at central auction prices $r$ to optimal solutions of the RESERVE-LP(r) for $r$ and $\Omega$ and hence show the equality of the demands for these two mediator definitions for gross-substitute valuations of the bidders. Recall that an OR-player over gross-substitute valuations has a gross-substitute valuation [16]. Thus in this case we can regard the EF-mediator as having a gross-substitute valuation. This implies that a Walrasian equilibrium for the central auction exists and, with the efficient demand oracle defined above, can be computed efficiently when all bidders have gross-substitute valuations and all mediators are EF-mediators.

### 4.3 Computing an Equilibrium

The basic three-party auction is simple: First run the central auction at the exchange, then the local auctions at the mediators. In this section we summarize the details and analyze the time needed to compute a three-party competitive equilibrium. We assume that all bidders have gross-substitute valuations and that their valuations can be accessed via a demand oracle. We assume, for simplicity, that there are $m$ EF-mediators, each with $n/m$ distinct bidders. We will use known polynomial-time auctions for the two-party allocation problem, see [22] for a recent survey. Theorem 4 shows how such an auction can be modified to yield a RESERVE-WE(r) instead of a Walrasian equilibrium.

Let $A$ be a polynomial-time algorithm that can access $n$ gross-substitute valuations over $k$ items $\Omega$ via a demand oracle and outputs a Walrasian price vector $p \in \mathbb{R}^k$ and a Walrasian allocation $\langle \Omega_i \rangle_{i\in[n]}$. Let the runtime of $A$ be $T(n,k) = O(n^\alpha k^\beta)$ for constants $\alpha$, $\beta$.

Although we can assume oracle access to the bidders' valuations, we cannot assume it for the mediators' (gross-substitute) valuations, as they are not part of the input. However, as outlined in the previous section, a mediator can determine a set in her demand by running a single virtual auction to compute a RESERVE-WE(r), i.e., there is an efficient demand oracle for the mediators. Hence, solving the allocation problem for the central auction can be done in time $T(m,k) \cdot T(n/m,k) = O(n^\alpha k^{2\beta})$. Further, the local auctions for all mediators take time $O(m \cdot T(n/m,k))$ and thus the total time to compute a three-party competitive equilibrium is $O(n^\alpha k^{2\beta})$.

Note that the computation at the exchange takes only $T(m,k)$ time and that the mediators are assumed to be separated, that is, the computation at the mediators can be done in parallel.

In the context of ad exchanges it is natural to assume that the number of items is very small and independent of the number of bidders. We discuss the computation of an equilibrium in this case in Appendix F.

## 5 Short Discussion

We proposed a new model for auctions at ad exchanges. Our model is more general than previous models in the sense that it takes the incentives of all three types of participants into account and that it allows to express preferences over multiple items. Interestingly, at least when gross-substitute valuations are considered, this generality does not come at the cost of tractability, as shown by our polynomial-time algorithm. Note that this is the most general result we could expect in light of the classical (two-sided) literature on combinatorial auctions.[10]

**Acknowledgment.** We wish to thank Noam Nisan for helpful discussions.

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[10] We elaborate and sketch new directions in Appendix A.

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# A Further Discussion and Future Directions

We considered the special case of a small number of items for which we showed that existing polynomial-time algorithms for two-party equilibria can be sped-up by adding mediators.

In our model, with gross-substitute bidders, the revenue of the mediators only comes from decreasing the central seller's profit (stated formally in Appendix E). This explains the willingness of the bidders to use mediators. The central seller experiences a decrease in its workload with the introduction of mediators, which may partially describe its own inclination for participating in the market.

Since our model tries to capture a single "user-impression-of-a-web-page" sold at an ad exchange, a natural follow up work will try to model what happens over time. This direction should take into account the change in the environment, as the bidders and their valuations as well as the mediators and their connections to bidders can be different for each user impression.

# B Proofs of Theorems on "Model and Equilibrium", Section 3

*Proof (Proof of Theorem 1).* A Walrasian equilibrium in the central auction is a price vector $r \geq 0$ and an allocation $\langle \Omega_i \rangle$ of items to mediators such that every item with strictly positive price is allocated to a mediator and every mediator is allocated a set in her demand $D_i(r)$. By the definition of $D_i(r)$, the virtual auction for every set $S \in D_i(r)$ computes an allocation of the items in $S$ to her bidders and envy-free prices $p^i \geq r$ (on $\Omega$) such that every bidder in $\mathcal{B}_i$ is allocated a set in his demand at prices $p^i$ and $p^i_j = r_j$ for all items $j \notin S$. Thus all requirements of a three-party competitive equilibrium are satisfied.

*Proof (Proof of Theorem 2).* A Walrasian equilibrium is a price vector $r \geq 0$ and an allocation $\langle \Omega_b \rangle$ of items to bidders such that every bidder is envy-free and all items with non-zero price are allocated to a bidder. This equilibrium induces a trivial three-party competitive equilibrium where all price vectors are identical to $r$ and the allocation to mediators is uniquely determined by the allocation $\langle \Omega_b \rangle$ to bidders. To see this note that the allocation $\langle \Omega_b \rangle$ with prices $r$ is globally envy-free for all bidders and thus for a mediator $\mathcal{M}_i$ the minimal locally envy-free prices $\geq r$ are equal to $r$ for the set of items allocated to $\mathcal{B}_i$. The revenue of all mediators under this equilibrium is zero and for each mediator the set allocated to her is in her demand. [11]

[11] The above proof also holds for any other mediator definition that prohibits mediators to gain other revenue than from the competition between her bidders in the local

## C Proof of Theorem 4

$$\textsc{reserve-lp}(r): \text{maximize} \sum_{b\in\mathcal{B},\, S\subseteq\Omega} x_{b,S} v_b(S) + \sum_{j\in\Omega}\left(1 - \sum_{b\in\mathcal{B},\, S|j\in S} x_{b,S}\right) r_j$$
$$\text{subject to} \sum_{b\in\mathcal{B},\, S|j\in S} x_{b,S} \le 1 \quad \forall j\in\Omega$$
$$\sum_{S\subseteq\Omega} x_{b,S} \le 1 \quad \forall b\in\mathcal{B}$$
$$x_{b,S} \ge 0 \quad \forall b\in\mathcal{B},\, S\subseteq\Omega$$

We now show how the RESERVE-LP(r) is obtained. It is well known that for any collection $\{v\}$ of valuations a Walrasian equilibrium (WE) exists if and only if the linear programming relaxation of the welfare maximization problem (WELFARE-LP), given below, has an integral solution. The integral solution combined with optimal dual prices yields a Walrasian equilibrium and vice versa (see e.g. [4] for monotone valuations and [18] for more general valuations).

$$\text{maximize} \sum_{b\in\mathcal{B},\, S\subseteq\Omega} x_{b,S} v_b(S) \qquad (\textsc{welfare-lp})$$
$$\text{subject to} \sum_{b\in\mathcal{B},\, S|j\in S} x_{b,S} \le 1 \qquad \forall j\in\Omega \qquad (1)$$
$$\sum_{S\subseteq\Omega} x_{b,S} \le 1 \qquad \forall b\in\mathcal{B} \qquad (2)$$
$$x_{b,S} \ge 0 \qquad \forall b\in\mathcal{B},\, S\subseteq\Omega \qquad (3)$$

The dual is as follows.

$$\text{minimize} \sum_{b\in\mathcal{B}} u_b + \sum_{j\in\Omega} p_j \qquad (4)$$
$$\text{subject to } u_b + \sum_{j\in S} p_j \ge v_b(S) \qquad \forall b\in\mathcal{B},\, S\subseteq\Omega \qquad (5)$$
$$u_b \ge 0,\, p_j \ge 0 \qquad \forall b\in\mathcal{B},\, j\in\Omega \qquad (6)$$

We will think of the dual variables $p_j$s as prices of items and of $u_b$s as maximum utilities for the bidders. Note that the dual objective is a function of the $p$s as the $u$s are determined by them. Now consider the effect of reserve prices, i.e., for

auction. This is because there is no competition in the local auction when the allocation and prices in the central auction are determined by a Walrasian equilibrium between bidders and items.

all $j \in \Omega$ a lower bound $r_j \geq 0$ for the dual variables $p_j$.

$$\begin{aligned} \text{minimize} \quad & \sum_{b \in \mathcal{B}} u_b + \sum_{j \in \Omega} p_j && && (7) \\ \text{subject to} \quad & u_b + \sum_{j \in S} p_j \geq v_b(S) && \forall b \in \mathcal{B},\, S \subseteq \Omega && (8) \\ & u_b \geq 0,\, p_j \geq r_j && \forall b \in \mathcal{B},\, j \in \Omega && (9) \end{aligned}$$

We can reformulate this linear program by a variable transformation with $q_j = p_j - r_j$ for all $j \in \Omega$. The term $\sum_{j \in \Omega} r_j$ is part of the input and thus can be omitted from the objective value.

$$\begin{aligned} \text{minimize} \quad & \sum_{b \in \mathcal{B}} u_b + \sum_{j \in \Omega} q_j + \sum_{j \in \Omega} r_j && && (10) \\ \text{subject to} \quad & u_b + \sum_{j \in S} q_j \geq v_b(S) - \sum_{j \in S} r_j && \forall b \in \mathcal{B},\, S \subseteq \Omega && (11) \\ & u_b \geq 0,\, q_j \geq 0 && \forall b \in \mathcal{B},\, j \in \Omega && (12) \end{aligned}$$

With this reformulation we obtain the following primal, which we call RESERVE-LP(r). Again $\sum_{j \in \Omega} r_j$ can be omitted from the objective value without changing the set of solutions.

$$\begin{aligned} \text{maximize} \quad & \sum_{b \in \mathcal{B},\, S \subseteq \Omega} x_{b,S} \left( v_b(S) - \sum_{j \in S} r_j \right) + \sum_{j \in \Omega} r_j && && (\text{RESERVE-LP}(r)) \\ \text{subject to} \quad & \sum_{b \in \mathcal{B},\, S | j \in S} x_{b,S} \leq 1 && \forall j \in \Omega && (13) \\ & \sum_{S \subseteq \Omega} x_{b,S} \leq 1 && \forall b \in \mathcal{B} && (14) \\ & x_{b,S} \geq 0 && \forall b \in \mathcal{B},\, S \subseteq \Omega && (15) \end{aligned}$$

The objective value of the RESERVE-LP(r) can be rewritten as

$$\sum_{b \in \mathcal{B},\, S \subseteq \Omega} x_{b,S} v_b(S) + \sum_{j \in \Omega} \left( 1 - \sum_{b \in \mathcal{B},\, S | j \in S} x_{b,S} \right) r_j \,. \tag{16}$$

In the following proof we use the additional additive player, Definition 8.

*Proof (Proof of Theorem 4).* (a1) $\Rightarrow$: Let $\langle \Omega_b \rangle$ and prices $p$ be a RESERVE-WE(r) for bidders $\mathcal{B}$. Then $\langle \Omega_b \rangle$ is an envy-free allocation at prices $p \geq r$, and all unallocated items $j$ have price $p_j = r_j$. Let $\Omega_0$ denote the set of unallocated items. A WE for the bidders $\mathcal{B}'$ is given by prices $p$ and allocation $\langle \Omega'_{b'} \rangle$ with $\Omega'_{b'} = \Omega_{b'}$ for $b' \neq a$ and $\Omega'_a = \Omega_0$. All items are allocated in $\langle \Omega'_{b'} \rangle$. The allocation for the bidders $b' \neq a$ clearly is envy-free as neither allocation nor prices were changed. Bidder $a$ is envy-free because $p \geq r$ and $p_j = r_j$ for $j \in \Omega'_a$.

$\Leftarrow$: Let $\langle \Omega'_{b'} \rangle$ and prices $p'$ be a WE for the bidders $\mathcal{B}'$. Then $\langle \Omega'_{b'} \rangle$ is an envy-free allocation and all unallocated items have a price of zero. For bidder $a$ to be envy-free it must hold that all items $j$ not allocated to $a$ have a price $p_j \geq r_j$ and all items $j'$ allocated to $a$ have $p_{j'} \leq r_{j'}$. We construct a RESERVE-LP(r) for the bidders $\mathcal{B}$ as follows. For all items allocated to bidders in $\mathcal{B}$ in $\langle \Omega'_{b'} \rangle$ allocation and prices remain the same. For all other items $j$ their price is set to $r_j$ and they are left unallocated. The allocation for the bidders $\mathcal{B}$ remains envy-free because the prices of the now unallocated items were only increased.

(a2): First note that for bidders $\mathcal{B}'$ in a WE, and therefore in an integral solution to the WELFARE-LP, we can assume w.l.o.g. that all items are allocated because we have $r \geq 0$ and therefore all otherwise unallocated items can be allocated to the additive player $a$. The objective value of the WELFARE-LP for an integral solution $\langle \Omega'_{b'} \rangle$ for bidders $\mathcal{B}'$ can be written as $\sum_{b \in \mathcal{B}} v_b(\Omega_b) + r(\Omega'_a)$, which is, w.l.o.g., equal to $\sum_{b \in \mathcal{B}} v_b(\Omega_b) + r(\Omega \setminus \cup_{b \in \mathcal{B}} \Omega_b)$. The latter is equivalent to Equation (16) for the allocation $\langle \Omega_b \rangle$. Thus there is (w.l.o.g.) a one-to-one correspondence between integral solutions to the WELFARE-LP for bidders $\mathcal{B}'$ and integral solutions to the RESERVE-LP(r) for bidders $\mathcal{B}$. Hence, $\langle \Omega'_{b'} \rangle$ is an optimal solution to the WELFARE-LP for $\mathcal{B}'$ if and only if $\langle \Omega_b \rangle$ is an optimal solution to the RESERVE-LP(r) for $\mathcal{B}$. Note that the corresponding constraints are satisfied as both $\langle \Omega'_{b'} \rangle$ and $\langle \Omega_b \rangle$ are allocations, respectively. To complete the proof, consider the following chain of "iff" statements.

$$\begin{aligned} (\langle \Omega_b \rangle, p) \text{ is a RESERVE-WE(r) for } \mathcal{B} &\iff (\langle \Omega'_b \rangle, p') \text{ is a WE for } \mathcal{B}' , \\ (\langle \Omega'_b \rangle, p') \text{ is a WE for } \mathcal{B}' &\iff (\langle \Omega'_b \rangle) \text{ solves WELFARE-LP for } \mathcal{B}' , \\ (\langle \Omega'_b \rangle) \text{ solves WELFARE-LP for } \mathcal{B}' &\iff (\langle \Omega_b \rangle) \text{ solves RESERVE-LP(r) for } \mathcal{B} , \\ &\text{and thus} \\ (\langle \Omega_b \rangle, p) \text{ is a RESERVE-WE(r) for } \mathcal{B} &\iff (\langle \Omega_b \rangle) \text{ solves RESERVE-LP(r) for } \mathcal{B} . \end{aligned}$$

(b1): The valuations $\{v\}$ of the bidders in $\mathcal{B}$ are gross substitute if and only if the valuations $\{v'\}$ of the bidders in $\mathcal{B}'$ are gross substitute, as the only difference between $\mathcal{B}$ and $\mathcal{B}'$ is the additive bidder $a$ whose value for an item $j$ is equal to its reserve price $r_j \geq 0$. Recall that every (non-negative) additive valuation is gross substitute. The claim then directly follows from (a1) and the existence of a WE for $\{v'\}$.

(b2): To show that the RESERVE-WE(r) price vectors form a complete lattice, we have to show that for any two RESERVE-WE(r) price vectors $p_1$ and $p_2$ the price vectors $\min(p_1, p_2)$ and $\max(p_1, p_2)$, where the min and the max is meant element-wise, are RESERVE-WE(r) price vectors as well. We will use (a1) and that for gross-substitute valuations WE price vectors form a complete lattice. The latter implies that for two WE price vectors $q'_1$ and $q'_2$, we have that $q'_{\min} = \min(q'_1, q'_2)$ and $q'_{\max} = \max(q'_1, q'_2)$ are WE price vectors as well. Recall the relation of $p'$ and $p$ in (a1), i.e., $p = \max(p', r)$. By (a1) we have that $q_{\min}$ and $q_{\max}$ are RESERVE-WE(r) price vectors. Let $q'_1 = p'_1$ and let $q'_2 = p'_2$. The claim follows from $\min(p_1, p_2) = q_{\min}$ and $\max(p_1, p_2) = q_{\max}$.

## D Proof of Theorem 3

We phrase all the statements in this section for gross-substitute valuations of the bidders, although they all hold as long as all minimal envy-free prices that respect the reserve prices are equal to the minimum RESERVE-WE(r) prices.

If the valuations of the bidders in $\mathcal{B}_i$ are all gross substitute, by Theorem 4 (b) a RESERVE-WE(r) with minimum prices exists for all reserve prices $r \geq 0$. We will use this several times in this section.

We start the proof of Theorem 3 with showing that the mediator computes in every virtual auction a RESERVE-WE(r) with minimum prices. The proof of this lemma is given in the next subsection.

**Lemma 1.** *If the valuations of an* EF*-mediator's bidders are gross substitute, then the* EF*-mediator computes minimum* RESERVE-WE$(r)$ *prices in her virtual auctions, i.e., items not allocated in a virtual auction have a price equal to their reserve price.*

This lemma implies that whenever for a set of items $S$ a virtual auction computes globally envy-free prices $p^S$, these prices have to be equal to the minimum RESERVE-WE(r) prices on $\Omega$.

**Corollary 1.** *If the valuation functions of all bidders $b \in \mathcal{B}_i$ are gross substitute, then for reserve prices $r \geq 0$ and all sets $S \subseteq \Omega$ such that $R_{i,r}(S) \neq -1$ the virtual auction prices $p^S$ are equal to $p^\Omega$ for all items in $\Omega$.*

It follows that an item $j$ with $p_j^\Omega > r_j$ must be in all sets with $R_{i,r}(S) \neq -1$ and thus in all demand representatives of the mediator. This implies that two sets $S$ and $S'$ with $R_{i,r}(S) \neq -1$ and $R_{i,r}(S') \neq -1$ can only differ in items $j$ with $p_j^\Omega = r_j$. Thus if for both $S$ and $S'$ all items are allocated in the virtual auction, then $R_{i,r}(S) = p^S(S) - r(S) = p^{S'}(S') - r(S') = R_{i,r}(S')$. Furthermore if for a set $S''$ with $R_{i,r}(S'') \neq -1$ an item $j \in S''$ with $r_j > 0$ is not allocated in the virtual auction, then $R_{i,r}(S'') < R_{i,r}(S)$. Hence, if for a set $S$ with $R_{i,r}(S) \neq -1$ all items are allocated in the virtual auction, then $R_{i,r}(S) = \max_{S'} R_{i,r}(S')$ and thus $S$ is in the demand of the mediator. Note that by Definition 6, if there are multiple RESERVE-WE(r) allocations on $S$ for the prices $p^S$, the mediator chooses the one that maximizes $\sum_{b \in \mathcal{B}_i} p^S(S_b)$, i.e., if the mediator can allocate all items in $S$, he will.

**Corollary 2.** *Assume that the valuation functions of all bidders $b \in \mathcal{B}_i$ are gross substitute. Let $r \geq 0$ be some reserve prices. If for some set $S$ with $R_{i,r}(S) \neq -1$ all items with strictly positive reserve price can be allocated in the virtual auction of the mediator, then $S$ is in $D_i(r)$.*

To completely characterize demand and allocation of the EF-mediator, we first show a useful technical result. We compare the minimum RESERVE-WE(r) prices for a set $T \subseteq \Omega$ with the minimum RESERVE-WE(r) prices for a subset $S \subseteq T$. For this we will use the following well-known result for Walrasian equilibria by Gul and Stacchetti [10] that by Theorem 4 (a1) also holds with reserve prices.

**Lemma 2 ([10]).** *Any Walrasian price vector combined with any Walrasian allocation yields a Walrasian equilibrium.*

**Corollary 3 (of Lemma 2 and Theorem 4).** *For $r \geq 0$ a* RESERVE-WE$(r)$ *price vector combined with any* RESERVE-WE$(r)$ *allocation yields a* RESERVE-WE$(r)$.

The following lemma shows that for suitable sets $S$ and $T$, the minimum prices in a RESERVE-WE(r) on $S$ are equal for items in $S$ to the corresponding prices in $T$. Part (a) of the lemma was shown for monotone gross-substitute valuations without reserve prices in [10].

**Lemma 3.** *Assume the valuation functions of all bidders $b \in \mathcal{B}_i$ are gross substitute. Let $T \subseteq \Omega$ be a set of items and let $S$ be a subset of $T$. For fixed reserve prices $r \geq 0$, let $(\langle T_b \rangle, p^T)$ be a* RESERVE-WE$(r)$ *with minimum prices on $T$ and let $(\langle S_b \rangle, p^S)$ be a* RESERVE-WE$(r)$ *with minimum prices on $S$. Then (a) $p_j^T \leq p_j^S$ for all $j \in S$ and (b) if $\cup_b T_b \subseteq S$, then $p_j^T = p_j^S$ for all $j \in S$ and $(\langle T_b \rangle, p^S)$ is a* RESERVE-WE$(r)$ *with minimum prices on $S$.*

*Proof.* (a) Let $V$ be the maximal valuation of any bidder, i.e., $\max_{b, T' \subseteq \Omega} v_b(T')$. Let $p'_j = p_j^S$ for $j \in S$ and let $p'_j = \max(V, r_j)$ for $j \in T \setminus S$. Then $(\langle S_b \rangle, p')$ is envy-free for all bidders on $T$ and $p' \geq r$. By Lemma 7 the prices $p^T$ are the minimum envy-free prices $\geq r$ on $T$. Thus $p_j^T \leq p'_j$ for all $j \in T$ and hence $p_j^T \leq p_j^S$ for all $j \in S$.

(b) If the set $S$ contains all items in $\cup_b T_b$, then the prices $p^T$ restricted to the set $S$ with the allocation $\langle T_b \rangle$ are a RESERVE-WE(r) on $S$. Thus by the minimality of the prices $p^S$, we have $p_j^T \geq p_j^S$ for all $j \in S$. Combined with (a) this shows $p_j^T = p_j^S$ for all $j \in S$.

The allocation $\langle T_b \rangle$ with prices $p^T$ restricted to $S$ are a RESERVE-WE(r) on $S$ and the prices $p^T$ restricted to the set $S$ are equal to the minimum RESERVE-WE(r) prices $p^S$ on $S$. Hence by Corollary 3 $(\langle T_b \rangle, p^S)$ is a RESERVE-WE(r) with minimum prices on $S$.

To characterize the demand of the mediator we further need that the maximum revenue the mediator can obtain is non-negative for all reserve prices $r \geq 0$. To compare the demand $D_i(r)$ of an EF-mediator to the demand of an OR- mediator, we further use that for *every* set in $D_i(r)$ all items with positive reserve price are allocated in the local auction.

**Lemma 4.** *Assume the valuation functions of all bidders $b \in \mathcal{B}_i$ are gross substitute and let $r \geq 0$ be any reserve price vector. (a) There exists a (potentially empty) set $S \subseteq \Omega$ such that the revenue $R_{i,r}(S)$ of an* EF*-mediator $\mathcal{M}_i$ is non-negative. (b) For a set $T \in D_i(r)$ with virtual auction allocation $\langle T_b \rangle$ all items $j \in T$ with $r_j > 0$ are allocated.*

*Proof.* (a) Let $(\langle \Omega_b \rangle, p)$ be the outcome of the virtual auction of an EF-mediator for $\Omega$. Take $S = \cup_b \Omega_b$. By Lemma 1 and Lemma 3 (b) $(\langle \Omega_b \rangle, p)$ is not only envy-free on $\Omega$ but further is a RESERVE-WE(r) with minimum prices for the virtual auction of an EF-mediator for the set $S$. Thus the mediator can allocate all items in $S$ in her virtual auction for $S$. Thus by $p \geq r$ the revenue $R_{i,r}(S) = \sum_b p(\Omega_b) - r(S) = p(S) - r(S)$ of the mediator for the set $S$ is non-negative.

(b) By (a) we have $R_{i,r}(T) \geq 0$ and thus $R_{i,r}(T) = \sum_b p^T(T_b) - r(T)$. Consider the set $T' = \cup_b T_b$. Assume by contradiction some items with $r_j > 0$ are not allocated in $\langle T_b \rangle$. By Lemma 1 $p_j^T = r_j$ for all items $j \in \Omega \setminus T'$. For the virtual auction prices $p^{T'}$ for $T'$ we have by definition $p_j^{T'} = r_j$ for $j \in \Omega \setminus T'$. By Lemma 3 (b) $p_j^{T'} = p_j^T$ for all $j \in T'$ and thus $p_j^{T'} = p_j^T$ for all $j \in \Omega$. Thus $(\langle T_b \rangle, p^{T'})$ is envy-free on the whole set of items $\Omega$, i.e., $R_{i,r}(T') \neq -1$. The mediator can allocate all items in $T'$; hence, $R_{i,r}(T') = p(T') - r(T') > R_{i,r}(T) = p(T') - r(T)$, a contradiction to $T \in D_i(r)$.

This proof gives us immediately an efficient way to determine a set $S$ with $R_{i,r}(S) \geq 0$: Run the virtual auction on $\Omega$ with reserve prices $r$ and return the set $S$ of allocated items. Combined with Corollary 1, this procedure actually yields not only a set with non-negative revenue but even a set in the demand of the mediator.

Before we continue, we observe a relation between the utility of the OR-player for reserve prices $r$ and the OR over modified valuation functions[12] $\{\widetilde{v}\}$ with $\widetilde{v}_b(S) = v_b(S) - r(S)$ for all $S \subseteq \Omega$. Note that $\widetilde{v}_{\text{OR}}(S) + r(S)$ equals the optimal value of the RESERVE-LP(r) on $S$ as long as an optimal integral solution exists. This relation gives a characterization of the demand of the OR-player with reserve prices.

**Observation 5** *The utility of the* OR*-player at prices $r$ is given by*

$$u_{\text{OR},r}(S) = \max_{\langle S_b \rangle} \left( \sum_{b \in \mathcal{B}_i} v_b(S_b) \right) - r(S) = \max_{\langle S_b \rangle} \left( \sum_{b \in \mathcal{B}_i} v_b(S_b) - r(S_b) \right) - r(S \setminus \cup_b S_b)$$

*The* OR *of the valuation functions $\widetilde{v}_b(S) = v_b(S) - r(S)$ is given by*

$$\widetilde{v}_{\text{OR}}(S) = \max_{\langle S_b \rangle} \sum_{b \in \mathcal{B}_i} \widetilde{v}_b(S_b) = \max_{\langle S_b \rangle} \left( \sum_{b \in \mathcal{B}_i} v_b(S_b) - r(S_b) \right)$$

*By definition we have $\widetilde{v}_{\text{OR}}(S) \geq u_{\text{OR},r}(S)$ (1).*
*Let the allocation $\langle S_b^* \rangle$ be $\arg\max_{\langle S_b \rangle} \sum_{b \in \mathcal{B}_i} \widetilde{v}_b(S_b)$ for the set $S$ and let $S^* = \cup_b S_b^* \subseteq S$. Then $\widetilde{v}_{\text{OR}}(S) = \widetilde{v}_{\text{OR}}(S^*) = u_{\text{OR},r}(S^*)$ (2a). Thus $\widetilde{v}_{\text{OR}}(S) > u_{\text{OR},r}(S)$ iff $u_{\text{OR},r}(S^*) > u_{\text{OR},r}(S)$ iff $S \notin D_{\text{OR}}(r)$ (2b).*

The following two lemmata finally show that the demand of an EF-mediator is equal to the demand of an OR-player for any central auctions prices $r \geq 0$ and gross-substitute valuations of the bidders. The proofs combine the results obtained so far to relate both demands to an optimal solution of the RESERVE-LP(r) for reserve prices $r$ and the items in $\Omega$.

[12] The valuations $\{\widetilde{v}\}$ might be non-monotone even if the valuations $\{v\}$ are monotone. This is not relevant here.

**Lemma 5.** *If the valuation functions of all bidders $b \in \mathcal{B}_i$ are gross substitute, then for any reserve prices $r \geq 0$ every set in the demand of an* EF*-mediator is in the demand of the* OR*-player. Additionally, for every set $S$ in the demand the* OR*-player could use the* EF*-mediator's allocation of the items in $S$ to the bidders in $\mathcal{B}_i$ to maximize her utility.*

*Proof.* Let $S$ be a set in the demand $D_i(r)$ of an EF-mediator $\mathcal{M}_i$ for some reserve prices $r \geq 0$. By Lemma 4 (a) there exists a set $S'$ with $R_{i,r}(S') \geq 0$, thus for $S$ in the demand we have $R_{i,r}(S) \geq R_{i,r}(S') \geq 0$. Let $(\langle S_b \rangle, p)$ be the outcome of the virtual auction of $\mathcal{M}_i$ for the set $S$. By Lemma 1 $(\langle S_b \rangle, p)$ is a RESERVE-WE(r) with minimum prices on the item set $S$. Since $R_{i,r}(S) \geq 0$, the allocation $\langle S_b \rangle$ is envy-free on $\Omega$. As the prices $p$ are RESERVE-WE(r) prices for $S$ and are extended with $p_j = r_j$ for $j \in \Omega \setminus S$, the allocation $\langle S_b \rangle$ and prices $p$ are also a RESERVE-WE(r) for the item set $\Omega$. Hence by Theorem 4 (a2) the allocation $\langle S_b \rangle$ is an integral solution to the RESERVE-LP(r) and thus maximizes the objective value of the RESERVE-LP(r) for both the item sets $S$ and $\Omega$. Since the value of the RESERVE-LP(r) only depends on the allocated sets, the two objective values are the same. Note that by the definition of $\widetilde{v}_{\text{OR}}$ in Observation 5 the objective value of the RESERVE-LP(r) is given by $\widetilde{v}_{\text{OR}}(\Omega) + r(\Omega)$ with $\widetilde{v}_{\text{OR}}(\Omega) = \max_{S' \in \Omega} \widetilde{v}_{\text{OR}}(S')$. Thus we have $\widetilde{v}_{\text{OR}}(S) = \widetilde{v}_{\text{OR}}(\Omega) = \max_{S' \in \Omega} \widetilde{v}_{\text{OR}}(S')$. By Lemma 4 (b) we can assume that all items $j \in S$ with $r_j > 0$ are allocated in $\langle S_b \rangle$, which implies that $\sum_{b \in \mathcal{B}_i} r(S_b) = r(S)$ and thus $\widetilde{v}_{\text{OR}}(S) = u_{\text{OR},r}(S)$. Since we have $\widetilde{v}_{\text{OR}}(S') \geq u_{\text{OR},r}(S')$ for all $S' \subseteq \Omega$ by Observation 5 (1), this implies $u_{\text{OR},r}(S) \geq \max_{S' \in \Omega} u_{\text{OR},r}(S')$. Since $S \subseteq \Omega$, it also holds that $u_{\text{OR},r}(S) \leq \max_{S' \in \Omega} u_{\text{OR},r}(S')$, implying that $u_{\text{OR},r}(S) = \max_{S' \in \Omega} u_{\text{OR},r}(S')$. Thus, $S$ is in the demand $D_{\text{OR}}(r)$ of the OR-player for reserve prices $r$ and the OR-player could use the allocation $\langle S_b \rangle$ to maximize her utility.

**Lemma 6.** *If the valuation functions of all bidders $b \in \mathcal{B}_i$ are gross substitute, then for any reserve prices $r \geq 0$ every set in the demand of the* OR*-player is in the demand of an* EF*-mediator. Additionally, for every set $S$ in the demand the* EF*-mediator could use the* OR*-player's allocation of the items in $S$ to the bidders in $\mathcal{B}_i$ to maximize his revenue.*

*Proof.* Let $S$ be a set in the demand $D_{\text{OR}}(r)$ of the OR-player for some reserve prices $r \geq 0$. Let $\langle S_b \rangle$ be the allocation of the OR-player for the set $S$. By Observation 5 (2b) we have $u_{\text{OR},r}(S) = \widetilde{v}_{\text{OR}}(S)$. Recall that $\widetilde{v}_{\text{OR}}(S) + r(S)$ is equal to the objective value of the RESERVE-LP(r) for the set $S$. Furthermore $\widetilde{v}_{\text{OR}}(S) = \widetilde{v}_{\text{OR}}(\Omega)$ because otherwise by Observation 5 (2a) there would be some allocation $\langle \Omega_b \rangle$ with $S' = \cup_b \Omega_b$ s.t. $\widetilde{v}_{\text{OR}}(\Omega) = \widetilde{v}_{\text{OR}}(S') = u_{\text{OR},r}(S')$ and thus the utility of the OR-player for the set $S'$ would be higher than for the set $S$, contradicting $S \in D_{\text{OR}}(r)$. Hence the allocation $\langle S_b \rangle$ of the OR-player is an integral solution to the RESERVE-LP(r) on $S$ as well as on $\Omega$. Let $p$ be the minimum RESERVE-WE(r) prices $p$ such that $(\langle S_b \rangle, p)$ is a RESERVE-WE(r) on $\Omega$. By Lemma 3 (b) we know that $(\langle S_b \rangle, p)$, with the prices $p$ restricted to $S$, is also a RESERVE-WE(r) with minimum prices on $S$. By Lemma 1 the virtual auction of an EF-mediator $\mathcal{M}_i$ for the set $S$ computes the same unique minimum prices $p$. Further $\widetilde{v}_{\text{OR}}(S) = u_{\text{OR},r}(S)$ implies that all items with strictly positive reserve price are allocated in $\langle S_b \rangle$. Thus $\mathcal{M}_i$ could allocate all items in $S$ with strictly positive reserve price by using the allocation $\langle S_b \rangle$. The allocation $\langle S_b \rangle$ is envy-free at prices $p$ on $S$. Thus the revenue of the mediator for the set $S$ is not set to $-1$. Hence by Corollary 2 the set $S$ is in the demand of the EF-mediator.

### D.1 Proof of Lemma 1

Lemma 1 is a corollary to the following, more general, lemma.

**Lemma 7.** *Consider all envy-free outcomes with prices $p \geq r$ for a set of valuations $\{v\}$ and reserve prices $r \geq 0$. If the valuations $\{v\}$ are gross substitute, then the minimum* RESERVE-WE$(r)$ *prices are minimum envy-free prices $p$ with $p \geq r$ among all envy-free outcomes with $p \geq r$.*

*Proof.* By Theorem 4 (b2) all RESERVE-WE(r) price vectors form a lattice. Let $p^*$ be the minimum price in this lattice. Recall that every RESERVE-WE(r) price vector is also an envy-free price vector. Assume by contradiction there exists an envy-free price $p$ such that $p^* \not\leq p$. Let $J = \{j \mid p_j < p^*_j\}$, let $\delta = \min_{j \in J} \{p^*_j - p_j\}$ be the *min-gap* and $p^{*-\delta J} = p^* - \delta_J$ where $\delta_J$ is the vector with value $\delta$ to each item in $J$ and 0 otherwise. Note that by assumption $J \neq \emptyset$, $\delta > 0$ and by minimality of $p^*$ no RESERVE-WE(r) allocation exists for $p^{*-\delta J}$.

Let $w_b(q)$ denote the maximum utility of a bidder $b$ for a price vector $q$, i.e., $w_b(q) = u_{b,q}(D)$ for some $D \in D_b(q)$. Following Gul and Stacchetti [11] and Ben-Zwi et al. [3], we define a requirement function and use Ben-Zwi et al. [3]'s extension of (one direction of) Hall's Theorem.

**Definition 9 (requirement function).** *Define for a set $S$, a bidder $b$, and prices $q$ the requirement function $f_{b,q}(S) = \min_{D \in D_b(q)}\{|D \cap S|\}$.*

**Observation 6 (compare Lemma 2.10 in [3])** *For a set $S$, a bidder $b$, and prices $q$, we have $f_{b,q}(S) \geq (w_b(q) - w_b(q + \delta_S))/\delta$.*

*Proof.* Let $D' = \arg\min_{D \in D_b(q)}\{|D \cap S|\}$. Then $w_b(q + \delta_S) \geq u_{b,q+\delta_S}(D') = u_{b,q}(D') - \delta f_{b,q}(S) = w_b(q) - \delta f_{b,q}(S)$, that is, $\delta f_{b,q}(S) \geq w_b(q) - w_b(q + \delta_S)$.

**Observation 7 (Observation 3.2 in [3])** *If for a price vector $q$ there exists $S$ such that $\sum_b f_{b,q}(S) > |S|$, then $q$ is not envy free. In this case we call $S$* over-demanded *at prices $q$.*

*Proof.* In any envy-free allocation of $S$, bidder $b$ must receive a set from his demand, thus $b$ must receive at least $f_{b,q}(S)$ many items of $S$. As each item of $S$ is allocated to at most one bidder, it follows that at least $\sum_b f_{b,q}(S) > |S|$ many items of $S$ are allocated in any envy-free allocation. Contradiction.

Note that the utilities $w_b(q)$ and prices $q$ are a feasible solution to the dual of the RESERVE-LP(r) for any prices with $q \geq r$. Further note that any optimal solution to the dual of the RESERVE-LP(r) implies that there exists a corresponding RESERVE-WE(r). By optimality of $p^*$ and the assumption that no RESERVE-WE(r) allocation exists for the prices $p^{*-\delta J}$, the objective value of the dual for $p^{*-\delta J}$ is strictly greater than the objective of the dual for $p^*$, i.e., $\sum_b w_b(p^{*-\delta J}) + p^{*-\delta J}(\Omega) > \sum_b w_b(p^*) + p^*(\Omega)$. Now by definition of $p^{*-\delta J}$ we know that $p^{*-\delta J}(\Omega)+\delta|J| = p^*(\Omega)$, hence together we have that $\sum_b w_b(p^{*-\delta J}) - \sum_b w_b(p^*) > \delta|J|$. With $\sum_b f_{b,p^{*-\delta J}}(J) \geq \left(\sum_b w_b(p^{*-\delta J}) - \sum_b w_b(p^*)\right)/\delta$ by Observation 6 we have that $\sum_b f_{b,p^{*-\delta J}}(J) > |J|$ and thus the set $J$ is over-demanded in $p^{*-\delta J}$ by Observation 7.

Next we use the following theorem by Gul and Stacchetti [11] to show that this implies $\sum_b f_{b,p}(J) > |J|$, i.e., a contradiction to the assumption that the prices $p$ are envy-free. By another result of Gul and Stacchetti [10], monotone valuations that are gross substitute also satisfy the single improvement property.

**Theorem 8 (Theorem 2 in [11]).** *Let $q^{(1)}, q^{(2)}$ be two price vectors such that $q^{(1)} \leq q^{(2)}$ and $S$ a set with $\forall j \in S,\ q^{(1)}(j) = q^{(2)}(j)$. Then for a bidder $b$ that fulfills the single improvement property the following apply*

1. $f_{b,q^{(1)}}(S) \leq f_{b,q^{(2)}}(S)$
2. $f_{b,q^{(1)}}(\Omega \setminus S) \geq f_{b,q^{(2)}}(\Omega \setminus S)$

Recall that by the definition of $J$ and $p^{*-\delta J}$, we have $p_j \leq p_j^{*-\delta J}$ for $j \in J$ and $p_j \geq p_j^{*-\delta J}$ for $j \notin J$. Now if we take $q^{(1)} = p^{*-\delta J}$ and take $q_j^{(2)} = p_j^{*-\delta J}$ if $j \in J$ and $q_j^{(2)} = p_j$ if $j \notin J$, then $q^{(1)} \leq q^{(2)}$ and thus by the first part of the theorem we get that $f_{b,q^{(2)}}(J) \geq f_{b,p^{*-\delta J}}(J) > |J|$, i.e., the set $J$ is over-demanded at prices $q^{(2)}$. On the other hand, if we take $q^{(3)} = p$ and the same $q^{(2)}$, then $q^{(3)} \leq q^{(2)}$ and $q_j^{(3)} = q_j^{(2)}$ for $j \notin J$, and hence by the second part of the theorem with $S = \Omega \setminus J$ and thus $\Omega \setminus S = J$ we have that $f_{b,p}(J) \geq f_{b,q^{(2)}}(J) > |J|$. This shows that the set $J$ is over-demanded in $p$ as well and thus there cannot be an envy-free allocation for prices $p$ by Observation 7.

# E Relation to Minimum Walrasian Prices

**Lemma 8.** *Let allocation $\langle \Omega_\beta \rangle$ and prices $q$ be a Walrasian equilibrium with minimum prices for gross-substitute bidders $\mathcal{B}$ and items $\Omega$. Let each bidder be connected to exactly one of $m$ EF-mediators and let $\mathcal{B}_i$ denote the set of bidders connected to mediator $\mathcal{M}_i$. Let $r, p^1, p^2, \ldots, p^m$ be the price vectors of a three-party equilibrium for the mediators and bidders and let $\langle \Omega_i \rangle$ be the equilibrium allocation of items to mediators and $\langle \Omega_b \rangle$ the equilibrium allocation of items to bidders. Let $r$ and $\langle \Omega_i \rangle$ be a Walrasian equilibrium with minimum prices for the mediators and let $p_j = \max_i(p_j^i)$ for all items $j$. Then $p = q$.*

*Proof.* As $\langle \Omega_b \rangle$ and $p$ form a Walrasian equilibrium for bidders $\mathcal{B}$ and items $\Omega$, we have $p \geq q$. Recall $p \geq r$. Further $\langle \Omega_\beta \rangle$, $\langle \Omega_i' = \cup_{\beta \in \mathcal{B}_i} \Omega_\beta \rangle$, and $r' = p^{1\prime} = \cdots =$

$p^{m\prime} = q$ form a three-party equilibrium (compare Theorem 2) and $\langle \Omega_i' \rangle$ and $q$ provide a Walrasian equilibrium for the mediators. Thus by the minimality of $r$ we have $q \geq r$. Assume by contradiction that there exists an item $j$ with $p_j > q_j$. By Lemma 1 and Corollary 1 each price vector $p^i$ is the minimum envy-free price vector $\geq r$ for the bidders $\mathcal{B}_i$. Thus at prices $q \not\geq p$ there is no allocation that is envy-free for all bidders $\mathcal{B}$, a contradiction.

## F Small Number of Items

In the context of ad exchange it is natural to assume that the number of items is very small and independent of the number of bidders. The results on this section will hold as long as the number of items $k = o(\log n)$.

When the number of items is that small, bidders' valuations can be represented as complete lists. More than that, given a bidder valuation oracle, it takes only $2^k$ queries to compile such a list. In order to find the valuation lists of all the mediators as well, we have to solve the allocation problem of each mediator $2^k$ times, i.e., compute the OR of the bidder valuations for all subsets. Given the valuations of the mediators, the central auction is equivalent to solving the two-party allocation problem for the mediators. Let $T'(n,k)$ be the runtime of algorithm $A$ when valuations are accessed via a valuation oracle. Then the overall running time to determine a three-party competitive equilibrium with this approach is $\widehat{T}(n,m,k) = m \cdot 2^k \cdot T'(n/m,k) + T'(m,k)$.

We show next how this approach can be extended to an almost linear time algorithm for such a small number of items by artificially introducing mediators of mediators (and recurse). Assume for simplicity $T'(n,k) = O(n^\alpha \cdot f(k))$ where $f(\cdot)$ is at most exponential in $k$ and $\alpha = 1+\gamma$ for some $\gamma > 0$.[13] By choosing $m = n^{1/2}$ we obtain a running time of

$$\begin{aligned}
\widehat{T}(n, n^{1/2}, k) &= n^{1/2} \cdot 2^k \cdot n^{\alpha/2} \cdot f(k) + n^{\alpha/2} \cdot f(k)\,, \\
&= \left(2^k n^{1+\gamma/2} + n^{1/2+\gamma/2}\right) \cdot f(k)\,, \\
&\leq c \cdot n^{\alpha/2+1/2} \cdot 2^{2k}\,,
\end{aligned}$$

for some constant $c \geq 0$. Let us add one level of recursion:

$$\begin{aligned}
\widehat{T}(n, n^{1/2}, k) &= 2^k \cdot n^{1/2} \cdot \widehat{T}(n^{1/2}, n^{1/4}, k) \\
&\quad + \widehat{T}(n^{1/2}, n^{1/4}, k)\,, \\
&\leq c \cdot 2^{3k} \cdot n^{1/2} \cdot (n^{1/2})^{\alpha/2+1/2} \\
&\quad + c \cdot 2^{2k} \cdot (n^{1/2})^{\alpha/2+1/2}\,, \\
&\leq c \cdot 2^{3k} \cdot (n^{1/2+1/4+\gamma/4+1/4} \\
&\quad + n^{1/4+\gamma/4+1/4})\,, \\
&\leq c \cdot 2^{3k} \cdot n^{(\alpha/2+1/2)/2+1/2}\,.
\end{aligned}$$

[13] Current methods have $\alpha = 6$ and thus $\gamma = 5$.

For $t$ levels of mediators we obtain $\widehat{T}(n, n^{1/2}, k) \leq c \cdot 2^{(t+1)k} \cdot n^{\alpha_t/2+1/2}$ where $\alpha_0 = \alpha$ and $\alpha_t = \frac{\alpha_{t-1}}{2} + \frac{1}{2} = \frac{\gamma}{2^t} + 1$. Since for constant $\delta = 1/(t+1)$ we have that $k = o(\log n)$ implies $k = o(\log n^\delta)$ and $\alpha$ is constant, we can choose $t$ to achieve a runtime of $O(n^{1+\varepsilon+o(1)})$ for any fixed $\varepsilon > 0$.

An almost linear time algorithm to solve the two-party allocation problem when $k = o(\frac{\log n}{\log \log n})$ can be obtained by reducing the problem to unit-demand valuations in the following way. Assume there are $n$ bidders and $k = o(\frac{\log n}{\log \log n})$ items $\Omega$. The following method computes in almost linear time an allocation between bidders and items that maximizes social welfare (i.e., $\sum_b v_b(\Omega_b)$), which is equal to a Walrasian allocation if it exists. Consider all possible partitions of the $k$ items from which there are $O(k^k) = O(2^{k \log k})$ many. For a partition $P$ let the sets in the partition be the new items and let the value of the bidders for a new item be their value for the set. Define a unit-demand valuation function for each bidder based on these values. Then solve the allocation problem for the new items and the unit-demand valuations. The resulting allocation maximizes social welfare for the given partition. Over all possible partitions the one with maximum social welfare yields the desired solution. For unit-demand valuations the allocation problem is equivalent to the maximum weight bipartite matching problem that can be solved with the Hungarian method in time $O(nk^2)$ [8]. Thus the total time is $O(nk^{k+2})$.