Subsampling Matrices for Wavelet Decompositions on Body Centered Cubic Lattices

Alireza Entezari, Torsten Möller and Jacques Vaisey
Simon Fraser University, Burnaby, BC, V5A 1S6, Canada
aentezar@cs.sfu.ca, torsten@cs.sfu.ca, vaisey@sfu.ca

***Abstract*— This article derives a family of dilation matrices for the body-centered cubic (BCC) lattice, which is optimal in the sense of spectral sphere packing. While satisfying the necessary conditions for dilation, these matrices are all cube roots of an integer scalar matrix. This property offers theoretical advantages for construction of wavelet functions in addition to the practical advantages when iterating through a perfect reconstruction filter bank based on BCC downsampling. Lastly, we factor the BCC matrix into two matrices that allow us to cascade two two-channel perfect reconstruction filter banks in order to construct a four-channel perfect reconstruction filter bank based on BCC downsampling.**

***Index Terms*— Dilation Matrix, Wavelet Decomposition, Body Centered Cubic Lattice**

I. Introduction

The sampling lattice determines the spectrum captured by multidimensional sampling. In three dimensions, the body-centered cubic (BCC) lattice [1] is known to allow for the minimum sampling density when the spectral support of the sampled signal is spherical. This result is due to the fact that the position of replicas in the frequency domain is determined by the face-centered cubic lattice (FCC), the reciprocal of the BCC lattice, which is the structure that produces the densest possible three dimensional packing structure. Theußl et al. [1], for example, have exploited the optimality of BCC sampling by resampling bandlimited signals from Cartesian to BCC lattices in order to obtain representations with fewer samples.

When used in multidimensional subsampling, efficient lattice structures allow for increased fidelity with a given number of samples. Nonetheless, the application of BCC subsampling to the wavelet transform has been thought to be impossible [2] due to the difficulty in specifying a suitable dilation matrix, $\boldsymbol{D}$, which corresponds to the subsampling factor in one-dimension. Here we overcome this difficulty and present a design method for determining families of admissible dilation matrices for multidimensional lattice structures. We then show how these matrices can be used to design three dimensional nonseparable perfect reconstruction filter banks, as in the wavelet transform.

II. The Sampling Matrix

A sampling lattice can be represented by a sampling matrix that gathers a set of basis vectors, indexing the lattice points, into columns. Note that this sampling matrix is not unique since a different set of basis vectors for the same sampling process yields a different sampling matrix. The sampling process is called separable if it can be represented with a diagonal matrix. The sampling matrix commonly used for representing the BCC lattice is [1]:

$$\boldsymbol{M} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \tag{1}$$

One can use this sampling matrix when resampling signals onto a BCC lattice; however, this matrix is not suitable for the subsampling operation (dilation), which involves setting $y(\boldsymbol{n}) = x(\boldsymbol{D}\boldsymbol{n})$, where $\boldsymbol{n}$ is an integer vector.

The role of the sampling matrix becomes crucial when used in dilation equation since iterating on the downsampling process (as in wavelet transform) amounts to a dilation with integer powers of the sampling matrix. Different dilation matrices lead to different regularity properties on the perfect reconstruction filter bank. Moreover, existence and smoothness of the wavelet basis functions highly depend on this matrix.

The dilation matrix $\boldsymbol{D}$ defined over the underlying lattice Γ must satisfy the following properties [3], [4]:

- $\boldsymbol{D}$ leaves Γ invariant, or $\boldsymbol{D}\Gamma \subset \Gamma$;
- all the eigenvalues, λ_i, of $\boldsymbol{D}$ must satisfy $|\lambda_i| > 1$.

If the input signal is sampled on a Cartesian (separable) lattice ($\Gamma = \mathbb{Z}^3$), then these properties confine an *admissible* subsampling matrix to be an integer matrix whose spectrum lies outside the closed unit disc.

Putting further restrictions on admissibility, it is of practical importance if we return to Γ after a relatively small number of downsampling iterations [2], [4], [5], since we will only have a small set of lattice structures in the multiresolution pyramid. This property implies that $\boldsymbol{D}$ be the n^{th} root of a scalar matrix; i.e., $\boldsymbol{D}^n = k\boldsymbol{I}, k \in \mathbb{Z}$ for some value of n. This property is also of theoretical importance since it significantly improves the construction of smooth wavelet functions as illustrated by Gröchenig [6].

A necessary condition for $\boldsymbol{D}$ to be a root of a scalar matrix is for the dilation matrix to have equal magnitude eigenvalues for $\boldsymbol{D}^n$ has equal eigenvalues [1]. This condition helps to preserve the geometric properties of the original signal in the lower resolutions since we obtain equal dilations along each of the eigenvectors: $|\lambda_1| = |\lambda_2| = |\lambda_3|$.

[1]If λ is an eigenvalue of $\boldsymbol{D}$ then λ^n is the corresponding eigenvalue of $\boldsymbol{D}^n$

Unfortunately, the matrix $\boldsymbol{M}$ in Equation 1 has $\lambda_1 = 2, \lambda_2 = -2$ and $\lambda_3 = 1$ and hence no dilation is obtained along the third eigenvector when this matrix is used for subsampling. Although many sampling matrices generate the same lattice, Cooklev [2] has hypothesized that the application of BCC subsampling to the wavelet transform is impossible in practice due to the difficulties associated with finding an admissible matrix. Specifically, the subsampling matrix reported in [2], has $\lambda_1 = -2$ and $\lambda_2 = \lambda_3 = \sqrt{2}$ and thus dilates unequally in three dimensions. Therefore, repeated subsampling will never result in a return to the original lattice.

Our approach to finding suitable $\boldsymbol{D}$ matrices is to derive additional characteristics for admissible matrices and to reduce the size of the search space of possible matrices by imposing structure. We start by using the fact that for a given sampling matrix $\boldsymbol{M}$, representing a lattice, every other matrix $\hat{\boldsymbol{M}}$, representing this same lattice can be obtained via $\hat{\boldsymbol{M}} = \boldsymbol{M}\boldsymbol{N}$ with $\boldsymbol{N}$ an integer matrix having $|\det \boldsymbol{N}| = 1$. This is a result known in point lattice theory [7] and can be easily proved by observing that if $\boldsymbol{M}$ and $\hat{\boldsymbol{M}}$ represent the same lattice, then every point on the span of $\boldsymbol{M}$ is in the span of $\hat{\boldsymbol{M}}$ and vice versa. Denoting $\boldsymbol{N} = \begin{pmatrix} \boldsymbol{n}_0 & \boldsymbol{n}_1 & \boldsymbol{n}_2 \end{pmatrix}$ and $\hat{\boldsymbol{N}} = \begin{pmatrix} \hat{\boldsymbol{n}}_0 & \hat{\boldsymbol{n}}_1 & \hat{\boldsymbol{n}}_2 \end{pmatrix}$ as the representation of the lattice base vectors in the other lattice ($\boldsymbol{n}_i$ and $\hat{\boldsymbol{n}}_i$ are thus integer vectors), allows us to write

$$\boldsymbol{M}\boldsymbol{I} = \hat{\boldsymbol{M}}\hat{\boldsymbol{N}} \tag{2}$$

$$\hat{\boldsymbol{M}}\boldsymbol{I} = \boldsymbol{M}\boldsymbol{N}, \tag{3}$$

By substituting (3) into (2) we have $\boldsymbol{M} = \boldsymbol{M}\boldsymbol{N}\hat{\boldsymbol{N}}$. Since $\boldsymbol{M}$ is nonsingular, we have $\boldsymbol{N}\hat{\boldsymbol{N}} = \boldsymbol{I}$ and because both matrices are integer we conclude that $|\det \boldsymbol{N}| = 1$.

Applying this result to Equation 1, we see that any dilation matrix representing the BCC lattice must satisfy $|\det \boldsymbol{D}| = |\det \boldsymbol{M}|$; hence, $\det \boldsymbol{D} = \pm 4$. A direct result of this observation is that for any matrix representing the BCC lattice at least three downsampling iterations are required to return to the original Cartesian lattice. In other words, the smallest value of n that can result in an integer solution for k in $\boldsymbol{D}^n = k\boldsymbol{I}$ is $n = 3$. This becomes apparent by taking the determinants on both sides of the equation $\boldsymbol{D}^n = k\boldsymbol{I}$. Since there are no integer solutions for k in $k^3 = (\pm 4)^n$ for $0 < n < 3$, we choose the smallest value of n for which an integer solution for k exists. This choice implies $n = 3$ and consequently $k = \det \boldsymbol{D}$.

It is also possible to put constraints on the eigenvalues of $\boldsymbol{D}$. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of $\boldsymbol{D}$ then $\lambda_1^3, \lambda_2^3, \lambda_3^3$ are the eigenvalues of $\boldsymbol{D}^3$. Since $\boldsymbol{D}^3 = (\det \boldsymbol{D})\boldsymbol{I}$, the eigenvalues of $\boldsymbol{D}$ must satisfy $\lambda_1^3 = \lambda_2^3 = \lambda_3^3 = \det \boldsymbol{D}$. We can conclude that the roots of $\lambda^3 = \det \boldsymbol{D}$ are valid eigenvalues of $\boldsymbol{D}$. Moreover, these eigenvalues make $\boldsymbol{D}$ an admissible matrix since $|\lambda_1| = |\lambda_2| = |\lambda_3| = 4^{1/3} > 1$.

In order to find an admissible dilation matrix $\boldsymbol{D}$, we need to find an integer matrix $\boldsymbol{N}$ with $|\det \boldsymbol{N}| = 1$ that properly transforms $\boldsymbol{M}$. To impose structure on the problem, we examined the class of upper-triangular matrices for $\boldsymbol{N}$. This selection was made since it is easy to ensure that $|\det \boldsymbol{N}| = 1$ in a triangular matrix and the number of free variables to solve for is fairly small. In the three-dimensional case, we set

$$\boldsymbol{N} = \begin{pmatrix} (-1)^i & a & b \\ 0 & (-1)^j & c \\ 0 & 0 & (-1)^k \end{pmatrix} \tag{4}$$

with $i, j, k \in \{0, 1\}$. It is easy to see that $|\det \boldsymbol{N}| = 1$ and thus $\boldsymbol{D} = \boldsymbol{M}\boldsymbol{N}$, with $\boldsymbol{M}$ as in (1), will generate the BCC lattice. Applying the constraint that the eigenvalues of $\boldsymbol{M}\boldsymbol{N}$ must be roots of $\lambda^3 = \det \boldsymbol{D}$ then allows us to solve for the unknown constants.

Each of the eight possibilities for (i, j, k) results in a family of solutions, all of which produce admissible $\boldsymbol{D}$ matrices that are cube roots of the scalar matrix $(\det \boldsymbol{D})\boldsymbol{I}$. For an integer matrix $\boldsymbol{D}$, there is a total of 24 solutions. While dilation with each of these matrices contracts (or stretches) the spectrum of the signal differently, similar partitioning of the spectrum of the signal can be achieved by designing filters with the proper support for each of these matrices.

Although we can argue that these matrices are theoretically equivalent, there might be practical advantages in using one over another. These matrices differ by the way they tile the space. Consequently their fundamental parallelepipeds [2] and coset structures differ from each other. Since the downsampling operation effectively replaces samples common to one fundamental parallelepiped with one sample, the downsampled value would be a good representative (predictor) for the original samples if the (Euclidean) distance between the coset points common to a fundamental parallelepiped is small.

An illustration of two fundamental parallelepipeds along with their coset points are shown in Figure 1. It is clear from the figure that not all solutions are equally desirable.

As an example of generating an admissible $\boldsymbol{D}$, we select the case in Figure 1a and obtain:

$$\boldsymbol{D} = \boldsymbol{M} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ -1 & 2 & 1 \end{pmatrix} \tag{5}$$

After applying this $\boldsymbol{D}$ once, we convert the original Cartesian lattice to a BCC lattice. After applying $\boldsymbol{D}$ a second time ($\boldsymbol{D}^2$), we obtain the samples on the transpose of the FCC lattice, which is the reciprocal of the BCC lattice [1]. We return to a Cartesian lattice after three downsampling iterations using $\boldsymbol{D}$.

III. Perfect Reconstruction Filter Banks and Wavelets

Nonseparable multidimensional regular perfect reconstruction filter banks have been studied by Cooklev [2] and Kovačević [4] that extend the standard techniques of filter design to two-channel multidimensional transforms. In our scenario however, we wish to subsample according to the BCC lattice defined by (5), where the number of channels is

[2] The fundamental parallelepiped is the (hyper)volume formed by the basis vectors of the sampling operation; i.e. the columns of the subsampling matrix.

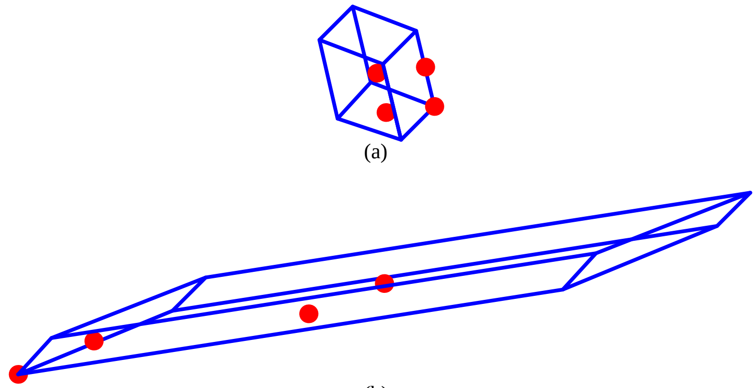

Fig. 1. Fundamental Parallelepipeds and their coset points corresponding to two admissible BCC dilation matrices $\boldsymbol{D} = \boldsymbol{M}\boldsymbol{N}$ as specified in Equation 1 and Equation 4. (a) A solution for $\boldsymbol{N}$ when $i = 1, j = 0, k = 1, a = 1, b = 0, c = 0$. (b) Another solution for $\boldsymbol{N}$ when $i = 0, j = 1, k = 0, a = 3, b = 8, c = -4$.

$|\det \boldsymbol{D}| = 4$ (i.e., the number of samples in the fundamental parallelepiped of the matrix).

However, designing a four-channel three-dimensional perfect-reconstruction filter bank is challenging due to the large size of the design space and the fact that the extension of spectral factorization methods to multiple dimensions is not straightforward.

As it turns out, we can factor (5) (or equivalently (1)) into two matrices, each having a determinant equal to two. This factorization implies that a BCC downsampling can be achieved by a 2D two-channel quincunx downsampling, followed by a two-channel FCC downsampling stage:

$$\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ -1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}$$

or

$$\boldsymbol{D} = \boldsymbol{Q}\boldsymbol{F}$$

Note that $|\det \boldsymbol{Q}| = |\det \boldsymbol{F}| = 2$.

This possibility has important implications, since it means that the four-channel PR filter bank can be implemented as a cascade of two two-channel PR filter banks. We can design multidimensional two-channel PR filter banks for each stage using existing methods. Let H_0^Q and H_1^Q be a pair of 2D lowpass and highpass filters that form a regular PR filter bank for the minor of q_{11} in $\boldsymbol{Q}$ and let H_0^F and H_1^F be a pair of 3D lowpass and highpass filters that constitute a regular PR filter bank for $\boldsymbol{F}$. Using the Noble identities, a regular PR filter bank for $\boldsymbol{D}$ can be derived [2]:

$$\begin{aligned} H_0^D(\omega_1, \omega_2, \omega_3) &= H_0^Q(\omega_2, \omega_3) H_0^F(\omega_1, \omega_2 + \omega_3, \omega_2 - \omega_3) \\ H_1^D(\omega_1, \omega_2, \omega_3) &= H_0^Q(\omega_2, \omega_3) H_1^F(\omega_1, \omega_2 + \omega_3, \omega_2 - \omega_3) \\ H_2^D(\omega_1, \omega_2, \omega_3) &= H_1^Q(\omega_2, \omega_3) H_0^F(\omega_1, \omega_2 + \omega_3, \omega_2 - \omega_3) \\ H_3^D(\omega_1, \omega_2, \omega_3) &= H_1^Q(\omega_2, \omega_3) H_1^F(\omega_1, \omega_2 + \omega_3, \omega_2 - \omega_3) \end{aligned}$$

The implications of this approach are that it is possible to design four-channel PR filter banks with any number of vanishing moments using the techniques developed by Cooklev and Kovačević [2], [4] for 3D two-channel PR filter banks.

However, this method will not yield the class of 3D four-channel filter banks that can*not* be factored in this way. Designing such filter banks is subject to further research as a suitable dilation matrix has now been found.

IV. Conclusions

This paper has presented a technique for designing admissible dilation matrices for nonseparable subsampling. The proposed method imposes structure on the set of possible solutions and then solves the resulting equations to produce a relatively small number of solutions for the BCC case. Different dilation matrices result in various subsampling structures, some of which may be more suitable than others. In the BCC case, we have also shown how to use the resulting dilation matrix to generate a four-channel PR filter bank.

Acknowledgments

We would like to thank the reviewers for constructive and stimulating comments. This research has been funded in part by the Natural Science and Engineering Council of Canada (NSERC) as well as British Columbia's Advanced Systems Institute (BC ASI).

References

[1] T. Theussl, T. Möller, and M. E. Gröller, "Optimal regular volume sampling," in *IEEE Visualization*, pp. 91–98, Oct 2001.

[2] T. Cooklev, *Regular perfect-reconstruction filter banks and wavelet bases*. Ph.D. dissertation, Tokyo University of Technology, Tokyo, Japan, 1995.

[3] K. Gröchenig and W. R. Madych, "Multiresolution analysis, Haar bases and self-similar tilings of $\mathbb{R}^n$," *IEEE Trans. on Information Theory*, vol. 38, pp. 556–568, Mar. 1992.

[4] J. Kovačević and M. Vetterli, "Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for $\mathscr{R}^n$," *IEEE Transactions on Information Theory*, vol. 17, pp. 939–954, Mar. 1992.

[5] M. Vetterli and J. Kovačević, *Wavelets and subband coding*. Signal Processing, Englewood Cliffs, NJ: Prentice Hall, 1995.

[6] K. Gröchenig and A. Ron, "Tight compactly supported wavelet frames of arbitrarily high regularity," *Proc. Amer. Math. Soc.*, no. 126, pp. 1101–1107, 1998.

[7] J. Cassels, *An Introduction to the Geometry of Numbers*. Springer-Verlag, 1997.