Mechanism Design by Creditability[⋆]

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Abstract. This paper attends to the problem of a mechanism designer seeking to influence the outcome of a strategic game based on her creditability. The mechanism designer offers additional payments to the players depending on their mutual choice of strategies in order to steer them to certain decisions. Of course, the mechanism designer aims at spending as little as possible and yet implementing her desired outcome. We present several algorithms for this optimization problem both for singleton target strategy profiles and target strategy profile regions. Furthermore, the paper shows how a bankrupt mechanism designer can decide efficiently whether strategy profiles can be implemented at no cost at all. Finally, risk-averse players and dynamic games are examined.

1 Introduction

Game theory is a powerful tool for analyzing decision making in systems with autonomous and rational (or selfish) participants. It is used in a wide variety of fields such as economics, politics, biology, or computer science. A major achievement of game theory is the insight that networks of self-interested agents often suffer from inefficiency due to effects of selfishness. Popular problems in computer science studied from a game theoretic point of view include *virus propagation* [1], *congestion* [2], or *network creation* [6], among many others.

If a game theoretic analysis reveals that a system suffers from the presence of selfish participants, mechanisms to encourage cooperation have to be devised. The field of *mechanism design* [5,9] is also subject to active research; for example, Cole et al. [3,4] have studied how incentive mechanisms can influence selfish behavior in a routing system.

In many distributed systems, a mechanism designer cannot change the rules of interactions. However, she may be able to influence the agents' behavior by offering payments for certain outcomes. On this account, Monderer and Tennenholtz [10] have initiated the study of a mechanism designer whose power is to some extent based on her monetary assets, primarily, though, on her *creditability*, i.e., the players trust her to pay the promised payments. Thus, a certain subset of outcomes is *implemented* in a given game if, by expecting additional non-negative payments, rational players will necessarily choose one of the desired outcomes. The designer faces the following optimization

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problem: How can a desired outcome be implemented at minimal cost? Surprisingly, it is sometimes possible to improve the performance of a given system merely by creditability, i.e., without any payments at all.

This paper extends [10] in various respects. First, an algorithm for finding an exact, incentive compatible implementation of a desired set of outcomes is given. We also show how a bankrupt mechanism designer can decide in polynomial time if a set of outcomes can be implemented at no costs at all, and an interesting connection to best response graphs is established. We propose and analyze efficient heuristic algorithms and demonstrate their performance. Furthermore, we extend our analysis for risk-averse behavior and study dynamic games where the mechanism designer offers payments in each round.

2 Model

Game Theory A *strategic game* can be described by a tuple $G = (N, X, U)$, where $N = \{1, 2, \dots, n\}$ is the set of *players* and each Player $i \in N$ can choose a *strategy* (action) from the set X_i. The product of all the individual players' strategies is denoted by $X := X_1 \times X_2 \times \ldots \times X_n$. In the following, a particular outcome $x \in X$ is called *strategy profile* and we refer to the set of all other players' strategies of a given Player i by $X_{-i} = X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n$. An element of X_i is denoted by x_i, and similarly, $x_{-i} \in X_{-i}$; hence x_{-i} is a vector consisting of the strategy profiles of x_i. Finally, $U = (U_1, U_2, \dots, U_n)$ is an n-tuple of *payoff functions*, where $U_i : X \to \mathbb{R}$ determines Player i's payoff arising from the game's outcome. Let $x_i, x'_i \in X_i$ be two strategies available to Player i. We say that x_i *dominates* x'_i iff $U_i(x_i, x_{-i}) \geq U_i(x'_i, x_{-i})$ for every $x_{-i} \in X_{-i}$ and there exists at least one x_{-i} for which a strict inequality holds. x_i is the *dominant* strategy for Player i if it dominates every other strategy $x'_i \in X_i \backslash \{x_i\}$. x_i is a *non-dominated* strategy if no other strategy dominates it. By $X^* = X^*_1 \times \ldots \times X^*_n$ we will denote the set of non-dominated strategy profiles, where X^*_i is the set of non-dominated strategies available to the individual Player i. The set of *best responses* $B_i(x_{-i})$ for Player i given the other players' actions is defined as $B_i(x_{-i}) := \{x_i | U_i(x_i, x_{-i}) = \max_{x_j \in X_i \backslash \{x_i\}} U_i(x_j, x_{-i})\}$. A *Nash equilibrium* is a strategy profile $x \in X$ such that for all $i \in N$, $x_i \in B_i(x_{-i})$.

Mechanism Design by Creditability This paper acts on the classic assumption that players are rational and always choose a non-dominated strategy. Additionally, it is assumed that players do not cooperate. We examine the impact of payments to players offered by a *mechanism designer* (an interested third party) who seeks to influence the outcome of a game. These payments are described by a tuple of non-negative payoff functions $V = (V_1, V_2, \dots, V_n)$, where $V_i : X \to \mathbb{R}^+$, i.e. the payments depend on the strategy Player i selects as well as on the choices of all other players. Thereby, we assume that the players trust the mechanism designer to finally pay the promised amount of money, i.e., consider her trustworthy (*mechanism design by creditability*). The original game $G = (N, X, U)$ is modified to $G(V) := (N, X, [U + V])$ by these payments, where $[U + V]_i(x) = U_i(x) + V_i(x)$, that is, each Player i obtains the payoff of V_i in addition to the payoffs of U_i. The players' choice of strategies changes accordingly: Each player now selects a non-dominated strategy in $G(V)$. Henceforth, the set of non-dominated strategy profiles of $G(V)$ is denoted by $X^*(V)$. A *strategy profile set* – also called *strategy profile region* – $O \subseteq X$ of G is a subset of all strategy profiles X, i.e., a region in the payoff matrix consisting of one or multiple strategy profiles. Similarly to X_i and X_{-i}, we define $O_i := \{x_i | \exists x_{-i} \in X_{-i} \text{ s.t. } (x_i, x_{-i}) \in O\}$ and $O_{-i} := \{x_{-i} | \exists x_i \in X_i \text{ s.t. } (x_i, x_{-i}) \in O\}$.

The mechanism designer's main objective is to force the players to choose a certain strategy profile or a set of strategy profiles. For a desired strategy profile region O, we say that payments V *implement* O if $\emptyset \subset X^*(V) \subseteq O$. V is called a *k-implementation* if, in addition $\sum_{i=1}^n V_i(x) \leq k$, $\forall x \in X^*(V)$. That is, the players' non-dominated strategies are within the desired strategy profile, and the payments do not exceed k for any possible outcome. Moreover, V is an *exact k-implementation* of O if $X^*(V) = O$ and $\sum_{i=1}^n V_i(x) \leq k \; \forall x \in X^*(V)$. The *cost* $k(O)$ of implementing O is the lowest of all non-negative numbers q for which there exists a q-implementation. If an implementation meets this lower bound, it is optimal, i.e., V is an *optimal implementation* of O if V implements O and $\max_{x \in X^*(V)} \sum_{i=1}^n V_i(x) = k(O)$. The cost $k^*(O)$ of implementing O exactly is the smallest non-negative number q for which there exists an exact q-implementation of O. V is an *optimal exact implementation* of O if it implements O exactly and requires cost $k^*(O)$. The set of all implementations of O will be denoted by $\mathcal{V}(O)$, and the set of all exact implementations of O by $\mathcal{V}^*(O)$. Finally, a strategy profile region $O = \{z\}$ of cardinality one – consisting of only one strategy profile – is called a *singleton*. Clearly, for singletons it holds that non-exact and exact k-implementations are equivalent. For simplicity's sake we often write z instead of $\{z\}$ and $V(z)$ instead of $\sum_{i \in N} V_i(z)$. Observe that only subsets of X which are in $2^{X_1} \times 2^{X_2} \times \ldots \times 2^{X_n} \subset 2^{X_1 \times X_2 \times \ldots \times X_n}$ can be implemented exactly. We call such a subset of X a *convex strategy profile region*.[1]

3 Algorithms and Analysis

3.1 Exact Implementation

Algorithm and Complexity Recall that in our model each player classifies the strategies available to her as either dominated or non-dominated. Thereby, each dominated strategy $x_i \in X_i \backslash X_i^*$ is dominated by at least one non-dominated strategy $x_i^* \in X_i^*$. In other words, a game determines for each Player i a relation M_i^G from dominated to non-dominated strategies $M_i^G : X_i \backslash X_i^* \to X_i^*$, where $M_i^G(x_i) = x_i^*$ states that $x_i \in X_i \backslash X_i^*$ is dominated by $x_i^* \in X_i^*$. See Fig. 1 for an example.

When implementing a strategy profile region O exactly, the mechanism designer creates a modified game $G(V)$ with a new relation $M_i^V : X_i \setminus O_i \to O_i$ such that all strategies outside O_i map to at least one strategy in O_i. Therewith, the set of all newly non-dominated strategies of Player i must constitute O_i. As every $V \in \mathcal{V}^*(O)$ determines a set of relations $M^V := \{M_i^V : i \in N\}$, there must be a set M^V for every V implementing O optimally as well. If we are given such an optimal relation set M^V without the corresponding optimal exact implementation, we can compute a V with minimal payments and the same relation M^V, i.e., given an optimal relation we can find an optimal exact implementation. As an illustrating example, assume an optimal relation set for G with $M_i^G(x_{i1}^*) = o_i$ and $M_i^G(x_{i2}^*) = o_i$. Thus,

[1] These regions define a convex area in the n-dimensional hyper-cuboid, provided that the strategies are depicted such that all o_i are next to each other.

we can compute V such that o_i must dominate x^*_{i1} and x^*_{i2} in $G(V)$, namely, the condition $U_i(o_i, o_{-i}) + V_i(o_i, o_{-i}) \geq \max_{s \in (x^*_{i1}, x^*_{i2})}(U_i(s, o_{-i}) + V_i(s, o_{-i}))$ must hold $\forall o_{-i} \in O_{-i}$. In an optimal implementation, Player i is not offered payments for strategy profiles of the form $(\bar{o}_i, x_{-i})$ where $\bar{o}_i \in X_i \backslash O_i,\ x_{-i} \in X_{-i}$. Hence, the condition above can be simplified to $V_i(o_i, o_{-i}) = \max(0, \max_{s \in \{x^*_{i1}, x^*_{i2}\}}(U_i(s, o_{-i}))) - U_i(o_i, o_{-i})$. Let $S_i(o_i){:=}\{s \in X_i \backslash O_i | M^V_i(s) = o_i\}$ be the set of strategies where M^V corresponds to an optimal exact implementation of O. Then, an implementation V with $V_i(\bar{o}_i, x_{-i}) = 0$, $V_i(o_i, \bar{o}_{-i}) = \infty$ for any Player i, and $V_i(o_i, o_{-i}) = \max\left\{0, \max_{s \in S_i(o_i)}(U_i(s, o_{-i}))\right\} - U_i(o_i, o_{-i})$ is an optimal exact implementation of O as well. Therefore, the problem of finding an optimal exact implementation V of O corresponds to the problem of finding an optimal set of relations $M^V_i : X_i \backslash O_i \to O_i$.

Our algorithm $\mathcal{ALG}_{exact}$ (cf. Algorithm 1) exploits this fact and constructs an implementation V for all possible relation sets, checks the cost that V would entail and returns the lowest cost found.

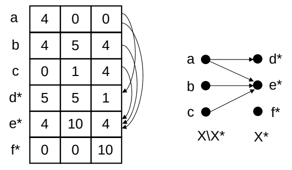

Fig. 1. A single player's game's view and its domination relation M^G.

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Algorithm 1 Exact k-Implementation ($\mathcal{ALG}_{exact}$)
Input: Game $G$, convex region $O$ with $O_{-i} \subset X_{-i} \forall\ i$
Output: $k^*(O)$
$V_i(x) := 0$, $W_i(x) := 0\ \forall x \in X\ ,\ i \in N$;
$V_i(o_i, \bar{o}_{-i}) := \infty\ \forall i \in N$, $o_i \in O_i\ ,\ \bar{o}_{-i} \in X_{-i} \backslash O_{-i}$;
compute $X^*$;
return ExactK($V$, $n$);
ExactK($V$, $i$):
Input: payments $V$, current Player $i$
Output: $k^*(O)$ for $G(V)$
if $|X^*_i(V) \backslash O_i| > 0$ then
    $s :=$ any strategy in $X^*_i(V) \backslash O_i$; $k_{best} := \infty$;
    for all $o_i \in O_i$ do
        for all $o_{-i} \in O_{-i}$ do
            $W_i(o_i, o_{-i}){:=}\max(0, U_i(s, o_{-i}) -$
            $(U_i(o_i, o_{-i}) + V_i(o_i, o_{-i})))$;
        $k :=$ ExactK($V + W$, $i$);
        if $k < k_{best}$ then
            $k_{best} := k$;
        for all $o_{-i} \in O_{-i}$ do
            $W_i(o_i, o_{-i}) := 0$;
    return $k_{best}$;
if $i > 1$ return $ExactK(V, i-1)$;
else return $\max_{o \in O} \sum_i V_i(o)$;
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Algorithm 2 Exact 0-Implementation ($\mathcal{ALG}_{bankrupt}$)
Input: Game $G$, convex region $O$ with $O_{-i} \subset X_{-i}\ \forall i$
Output: $\top$ if $k^*(O) = 0$, $\bot$ otherwise
compute $X^*$;
for all $i \in N$ do
    for all $s \in X^*_i \backslash O_i$ do
        dZero := $\bot$;
        for all $o_i \in O_i$ do
            b := $\top$;
            for all $o_{-i} \in O_{-i}$ do
                b := b $\wedge$ $(U_i(s, o_{-i}) \leq U_i(o_i, o_{-i}))$;
            dZero := dZero $\vee$ b;
        if $\neg$ dZero then
            return $\bot$;
return $\top$;
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Theorem 1. *$\mathcal{ALG}_{exact}$ computes a strategy profile region's optimal exact implementation cost in time* $\mathrm{O}\big(|X|^2 \max_{i \in N}(|O_i|^{n|X^*_i \backslash O_i|-1}) + n|O| \max_{i \in N}(|O_i|^{n|X^*_i \backslash O_i|})\big)$.

Note that $\mathcal{ALG}_{exact}$ has a large time complexity. In fact, a faster algorithm for this problem, called *Optimal Perturbation Algorithm* has been presented in [10]. In a nutshell, this algorithm proceeds as follows: After initializing V similarly to our algorithm, the values of the region O in the matrix V are increased slowly for every Player i, i.e., by all possible differences between an agent's payoffs in the original game. The algorithm terminates as soon as all strategies in $X^*_i \setminus O_i$ are dominated. Unfortunately, this algorithm does not always return an optimal implementation. Sometimes, as we show in Appendix A of the full version, the optimal perturbation algorithm increases the values unnecessarily. In fact, we even conjecture that deciding whether an k-exact implementation exists is $\mathbf{NP}$-hard.

Conjecture 1. Finding an optimal exact implementation of a strategy region is **NP**-hard.

Bankrupt Mechanism Designers Imagine a mechanism designer who is broke. At first sight, it seems that without any money, she will hardly be able to influence the outcome of a game. However, this intuition ignores the power of creditability: a game can have 0-implementable regions.

Let V be an exact implementation of O with exact costs $k^*(O)$. It holds that if $k^*(O) = 0$, V cannot contain any payments larger than 0 in O. Consequently, for an region O to be 0-implementable exactly, any strategy s outside O_i must be dominated within the range of O_{-i} by a o_i, or there must be one o_i for which no payoff $U_i(s, o_{-i})$ is larger than $U_i(o_i, o_{-i})$. In the latter case, the strategy o_i can still dominate s by using a payment $V(o_i, x_{-i})$ with $x_{-i} \in X_{-i} \backslash O_{-i}$ outside O. Note that this is only possible under the assumption that $O_{-i} \subset X_{-i}\ \forall i \in N$.

$\mathcal{ALG}_{bankrupt}$ (cf. Algorithm 2) describes how a bankrupt designer can decide in polynomial time whether a certain region is 0-implementable. It proceeds by checking for each Player i if the strategies in $X_i^* \backslash O_i$ are dominated or "almost" dominated within the range of O_{-i} by at least one strategy inside O_i. If there is one strategy without such a dominating strategy, O is not 0-implementable exactly. On the other hand, if for every strategy $s \in X_i^* \backslash O_i$ such a dominating strategy is found, O can be implemented exactly without expenses.

Theorem 2. *Given a convex strategy profile region O where $O_{-i} \subset X_{-i}\ \forall i$, Algorithm $\mathcal{ALG}_{bankrupt}$ decides whether O has an exact 0-implementation in time $\mathrm{O}\Big(n\,|X|^2\Big)$.*

Best Response Graphs Best response strategies maximize the payoff for a player given the other players' decisions. For now, let us restrict our analysis to games where the sets of best response strategies consist of only one strategy for each $x_{-i}\ \forall i \in N$. Given a game G, we construct a directed *best response graph* $\mathcal{G}_G$ with vertices v_x for strategy profiles $x \in X$ iff x is a best response for at least one player, i.e., if $\exists i \in N$ such that $x_i \in B_i(x_{-i})$. There is a directed edge $e = (v_x, v_y)$ iff $\exists i \in N$ such that $x_{-i} = y_{-i}$ and $\{y_i\} = B_i(y_{-i})$. In other words, an edge from v_x to v_y, indicates that it is better to play y_i instead of x_i for a player if for the other players' strategies $x_{-i} = y_{-i}$. A strategy profile region $O \subset X$ has a *corresponding subgraph* $\mathcal{G}_{G,O}$ containing the vertices $\{v_x | x \in O\}$ and the edges which both start and end in a vertex of the subgraph. We say $\mathcal{G}_{G,O}$ has an *outgoing edge* $e = (v_x, v_y)$ if $x \in O$ and $y \notin O$. Note that outgoing edges are not in the edge set of $\mathcal{G}_{G,O}$. Clearly, it holds that if a singleton x's corresponding subgraph $\mathcal{G}_{G,\{x\}}$ has no outgoing edges then x is a *Nash equilibrium*. More generally, we make the following observation.

Theorem 3. *Let G be a game and $|B_i(x_{-i})| = 1\ \forall i \in N, x_{-i} \in X_{-i}$. If a convex region O has an exact 0-implementation, then the corresponding subgraph $\mathcal{G}_{G,O}$ in the game's best response graph has no outgoing edges.*

In order to extend best response graphs to games with multiple best responses, we modify the edge construction as follows: In the general best response graph $\mathcal{G}_G$ of a game G there is a directed edge $e = (v_x, v_y)$ iff $\exists i \in N$ s.t. $x_{-i} = y_{-i}$, $y_i \in B_i(y_{-i})$ and $|B_i(y_{-i})| = 1$.

Corollary 1. *Theorem 3 holds for arbitrary games.*

Note that Theorem 3 is a generalization of Monderer and Tennenholtz' Corollary 1 in [10]. They discovered that for a singleton x, it holds that x has a 0-implementation if and only if x is a Nash equilibrium. While their observation covers the special case of singleton-regions, our theorem holds for any strategy profile region. Unfortunately, for general regions, one direction of the equivalence holding for singletons does not hold anymore due to the fact that 0-implementable regions O must contain a player's best response to any o_{-i} but they need not contain best responses exclusively.

5 / 4	5 / 5	5 / 1
10 / 10	10 / 0	0 / 10

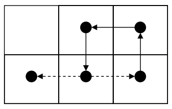

Fig. 2. Sample game G with best response graph $\mathcal{G}_G$. The Nash equilibrium in the bottom left corner has no outgoing edges. The dotted arrows do not belong to the edge set of $\mathcal{G}_G$ as the row has multiple best responses.

3.2 Non-Exact Implementation

In contrast to exact implementations, where the complete set of strategy profiles O must be non-dominated, the additional payments in non-exact implementations only have to ensure that a *subset* of O is the newly non-dominated region. Obviously, it matters which subset this is. Knowing that a subset $O' \subseteq O$ bears optimal costs, we could find $k(O)$ by computing $k^*(O')$. Apart from the fact that finding an optimal implementation includes solving the – believed to be $\mathbf{NP}$-hard – optimal exact implementation cost problem for at least one subregion of O, finding this subregion might also be $\mathbf{NP}$-hard since there are exponentially many possible subregions. In fact, a reduction from the SAT problem is presented in [10]. The authors show how to construct a 2-person game in polynomial time given a CNF formula such that the game has a 2-implementation if and only if the formula has a satisfying assignment. However, their proof is not correct: While there indeed exists a 2-implementation for every satisfiable formula, it can be shown that 2-implementations also exist for non-satisfiable formulas. E.g., strategy profiles $(x_i, x_i) \in O$ are always 1-implementable. Unfortunately, we were not able to correct their proof. However, we conjecture the problem to be $\mathbf{NP}$-hard, i.e., we assume that no algorithm can do much better than performing a brute force computation of the exact implementation costs (cf. Algorithm 1) of all possible subsets, unless $\mathbf{NP} = \mathbf{P}$.

Conjecture 2. Finding an optimal implementation of a strategy region is $\mathbf{NP}$-hard.

For the special case of zero cost regions, Theorem 3 implies the following result.

Corollary 2. *If a strategy profile region O has zero implementation cost then the corresponding subgraph $\mathcal{G}_{G,O}$ in the game's best response graph contains a subgraph $\mathcal{G}_{G,O'}$, $O' \subseteq O$, with no outgoing edges.*

Corollary 2 is useful to a bankrupt mechanism designer since searching the game's best response graph for subgraphs without outgoing edges helps her spot candidates for regions which can be implemented by mere creditability. In general though, the fact that finding optimal implementations seems computationally hard raises the question whether there are polynomial time algorithms achieving good approximations. As mentioned in Section 3.1, each V implementing a region O defines a domination relation $M_i^V : X_i \setminus O_i \to O_i$. This observation leads to the idea of designing heuristic algorithms that find a correct implementation by establishing a corresponding relation set $\{M_1, M_2, \ldots, M_n\}, M_i : X_i^* \backslash O_i \to O_i$ where each $x_i^* \in X_i^* \backslash O_i$ maps to at least one $o_i \in O_i$. These algorithms are guaranteed to find a correct implementation of O, however, the corresponding implementations may not be cost-optimal.

Our greedy algorithm $\mathcal{ALG}_{greedy}$ (cf. Algorithm 3) associates each strategy x_i^* yet to be dominated with the o_i with minimal distance $\varDelta_G$ to x_i^*, i.e., the maximum value that has to be added to $U_i(x_i', x_{-i})$ such that x_i' dominates x_i: $\varDelta_G(x_i, x_i') := \max_{x_{-i} \in X_{-i}} \max(0, U_i(x_i, x_{-i}) - U_i(x_i', x_{-i}))$. Similarly to the greedy approximation algorithm for the *set cover problem* [7,8] which chooses in each step the subset covering the most elements not covered already, $\mathcal{ALG}_{greedy}$ selects a pair of (x_i^*,o_i) such that by dominating x_i^* with o_i, the number of strategies in $X_i^* \backslash O_i$ that will be dominated therewith is maximal. Thus, in each step there will be an o_i assigned to dominate x_i^* which has minimal dominating cost. Additionally, $\mathcal{ALG}_{greedy}$ takes any opportunity to dominate multiple strategies. $\mathcal{ALG}_{greedy}$ is described in detail in Algorithm 3. It returns an implementation V of O; to determine V's cost, one needs to compute $\max_{x^* \in X^*(V)} \sum_{i \in N} V_i(x^*)$.

Theorem 4. *$\mathcal{ALG}_{greedy}$ returns an implementation of a convex strategy profile region $O \in X$ in time* $\mathrm{O}\big(n|X|^2 \max_{i \in N} |X_i^* \setminus O_i| + n|X| \max_{i \in N} |X_i^* \setminus O_i|^3\big)$.

$\mathcal{ALG}_{red}$ (cf. Algorithm 4) is a more sophisticated algorithm applying $\mathcal{ALG}_{greedy}$. Instead of terminating when the payment matrix V implements O, this algorithm continues to search for a payment matrix inducing even less cost. It uses $\mathcal{ALG}_{greedy}$ to approximate the cost repeatedly, varying the region to be implemented. As $\mathcal{ALG}_{greedy}$ leaves the while-loop if $X_i^*(V) \subseteq O_i$, it might miss out on cheap implementations where $X_i^*(V) \subseteq Q_i$, $Q_i \subset O_i$. $\mathcal{ALG}_{red}$ examines some of these subsets as well by calling $\mathcal{ALG}_{greedy}$ for some Q_i. If we manage to reduce the cost, we continue with $O_i := Q_i$ until neither the cost can be reduced anymore nor any strategies can be deleted from any O_i.

Theorem 5. *Let T_g denote the runtime of $\mathcal{ALG}_{greedy}$. $\mathcal{ALG}_{red}$ returns an implementation of O in time* $\mathrm{O}(n|O| \max_{i \in N} |O_i|(|O| + n + T_g))$.

Algorithm 3 Greedy Algorithm $\mathcal{ALG}_{greedy}$

```
Input: Game $G$, convex target region $O$
Output: Implementation $V$ of $O$
$V_i(x) := 0; W_i(x) := 0 \ \forall x \in X \ , i \in N$;
compute $X^*$;
for all $i \in N$ do
    $V_i(o_i, \bar{o}_{-i}) := \infty \ \forall o_i \in O_i \ , \bar{o}_{-i} \in X_{-i} \backslash O_{-i}$;
    while $X_i^*(V) \not\subseteq O_i$ do
        $c_{best} := 0; m_{best}$ :=null; $s_{best}$ :=null;
        for all $s \in X_i^*(V) \backslash O_i$ do
            $m := \arg\min_{o_i \in O_i}(\Delta_{G(V)}(s, o_i))$;
            for all $o_{-i} \in O_{-i}$ do
                $W_i(m, o_{-i})$:=max$(0, U_i(s, o_{-i}) -$
                $(U_i(m, o_{-i}) + V_i(m, o_{-i})))$;
            $c := 0$;
            for all $x \in X_i^* \setminus O_i$ do
                if $m$ dominates $x$ in $G(V + W)$ then
                    $c++$;
            if $c > c_{best}$ then
                $c_{best} := c$ ; $m_{best} := m$ ; $s_{best} := s$;
        for all $o_{-i} \in O_{-i}$ do
            $V_i(m_{best}, o_{-i})$+=max$(0, U_i(s_{best}, o_{-i}) -$
            $(U_i(m_{best}, o_{-i}) + V_i(m_{best}, o_{-i})))$;
return $V$;
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Algorithm 4 Reduction Algorithm $\mathcal{ALG}_{red}$

```
Input: Game $G$, convex target region $O$
Output: Implementation $V$ of $O$
$[k, V] := greedy(G, O)$;
$k_{temp} := -1$; $c_i := \perp \ \forall i$; $T_i := \{\} \ \forall i$;
while $(k > 0) \wedge (\exists i : |O_i| > 1) \wedge (\exists i : O_i \not\subseteq T_i)$ do
    for all $i \in N$ do
        $x_i := \arg\min_{o_i \in O_i} (\max_{o_{-i} \in O_{-i}} U_i(o_i, o_{-i}))$;
        if $(O_i \not\subseteq T_i) \wedge (\neg c_j \forall j) \wedge (x_i \in T_i)$ then
            $x_i$:=$\arg\min_{o_i \in O_i \backslash T_i} (\max_{o_{-i} \in O_{-i}} (U_i(o_i, o_{-i})))$;
        if $|O_i| > 1$ then
            $O_i := O_i \setminus \{x_i\}$;
        $[k_{temp}, V] := greedy(G, O)$;
        if $k_{temp} \geq k$ then
            $O_i := O_i \cup \{x_i\}$; $T_i := T_i \cup \{x_i\}$; $c_i := \perp$;
        else
            $k := k_{temp}$; $T_i := \{\} \ \forall i$; $c_i := \top$;
return $V$;
```

An alternative heuristic algorithm for computing a region O's implementation cost retrieves the region's cheapest singleton, i.e., $\min_{o \in O} k(o)$, where a singleton's implementation cost is $k(o) = \min_{o \in O} \sum_{i \in N} \max_{x_i \in X_i} (U_i(x_i, o_{-i}) - U_i(o_i, o_{-i}))$ [10]. The best singleton heuristic algorithm performs quite well for randomly generated games as our simulations reveal (cf. Section 4), but it can result in an arbitrarily large k in the worst case: Fig. 3 depicts a game where each singleton o in the region O consisting of the four bottom left profiles has cost $k(o) = 11$ whereas V implements O at cost 2.

$G =$

20, 0	11, 9	15, 15	15, 15
11, 9	20, 0	15, 15	15, 15
19, 10	10, 19	9, 11	0, 20
10, 19	19, 10	0, 20	9, 11

$V =$

0, ∞	0, ∞	0, 0	0, 0
0, ∞	0, ∞	0, 0	0, 0
1, 1	1, 1	∞, 0	∞, 0
1, 1	1, 1	∞, 0	∞, 0

Fig. 3. 2-player game where O's optimal implementation V yields a region $|X^*(V)| > 1$.

4 Simulation

All our algorithms return correct implementations of the desired strategy profile sets and – apart from the recursive algorithm $\mathcal{ALG}_{exact}$ for the optimal exact implementation –

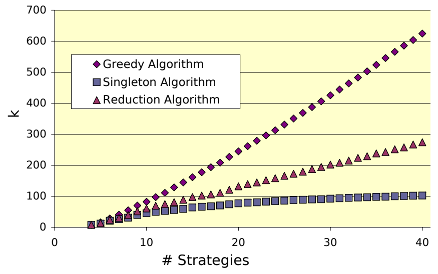
run in polynomial time. In order to study the quality of the resulting implementations, we performed several simulations comparing the implementation costs computed by the different algorithms. We have focused on two-person games using random game tables where both players have payoffs chosen uniformly at random from the interval $[0, max]$, for some constant max.

We can modify an implementation V of O, which yields a subset of O, without changing any entry $V_i(o), o \in O$, such that the resulting V implements O *exactly*.

Theorem 6. *If $O_{-i} \subset X_{-i}\ \forall i \in N$, it holds that $k^*(O) \leq \max_{o \in O} V(o)$ for an implementation V of O.*

Theorem 6 enables us to use $\mathcal{ALG}_{greedy}$ for an exact cost approximation by simply computing $\max_{o \in O} V(o)$ instead of $\max_{x \in X^*(V)} V(x)$.

Non-Exact Implementation We observe that implementing the best singleton often yields low costs. In other words, especially when large sets have to be implemented, our greedy algorithms tend to implement too many strategy profiles and consequently incur unnecessarily high costs. On average, the singleton algorithm performed much better than the other two, with $\mathcal{ALG}_{greedy}$ being the worst of the candidates. We presume that the $\mathcal{ALG}_{red}$ might improve relatively to the best singleton heuristic algorithm for larger player sets.



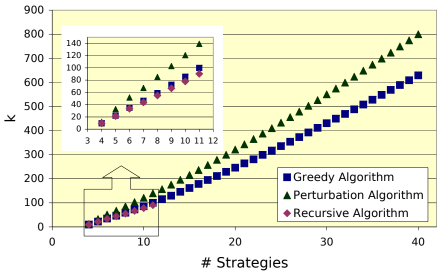


Fig. 4. The average implementation cost k of sets O over 100 random games where $|O_i| = \lfloor n/3 \rfloor$ (left: Non-exact, right: exact). The utility values are chosen uniformly at random from $[0, 20]$. For different intervals we obtain approximately the same result when normalizing k with the maximal possible value.

Exact Implementation Due to the large runtime of $\mathcal{ALG}_{exact}$, we were only able to compute k for a small number of strategies. However, for these cases, our simulations reveals that $\mathcal{ALG}_{greedy}$ often finds implementations which are close to optimal and is better than the perturbation algorithm. For different payoff value intervals $[0, max]$, we observe a faster increase in k than in the non-exact implementation case. This suggests that implementing a smaller region entails lower costs for random games on average.

Finally, we tested different options to choose the next strategy in $\mathcal{ALG}_{greedy}$ (Line 8) and $\mathcal{ALG}_{red}$ (Lines 5 and 7). However, none of the alternatives we tested performed better than the ones described in Section 3.

In conclusion, our simulations have shown that for the case of non-exact implementations, there are interesting differences between the algorithms proposed in Section 3. In particular, the additional reductions by $\mathcal{ALG}_{red}$ are beneficial. For the case of exact implementations, our modified greedy algorithm yields good results. As a final remark we want to mention that, although $\mathcal{ALG}_{greedy}$ and $\mathcal{ALG}_{red}$ may find cheap implementations in the average case, there are examples where the approximation ratio of these algorithms is large.

5 Variations

Mechanism design by creditability offers many interesting extensions. In this section, two alternative models of rationality are introduced. If we assume that players do not just select *any* non-dominated strategy, but have other parameters influencing their decision process, our model has to be adjusted. In many (real world) games, players typically do not know which strategies the other players will choose. In this case, a player cannot do better than assume the other players to select a strategy *at random*. If a player wants to maximize her gain, she will take the *average payoff* of strategies into account. This kind of decision making is analyzed in the subsequent section. Afterwards, risk-averse players are examined. Finally, we take a brief look at the dynamics of repeated games with an interested third party offering payments *in each round*.

5.1 Average Payoff Model

As a player may choose any non-dominated strategy, it is reasonable to compute the payoff which each of her strategy will yield *on average*. Thus, assuming no knowledge on the payoffs of the other players, each strategy x_i has an average payoff of $p_i(x_i) := \frac{1}{|X_{-i}|}\sum_{x_{-i}\in X_{-i}} U_i(x_i, x_{-i})$ for Player i. Player i will then select the strategy $s \in X_i$ with the largest $p_i(s)$, i.e., $s = \arg\max_{s\in X_i} p_i(s)$. If multiple strategies have the same average payoff, she plays one of them uniformly at random. For such average strategy games, we say that x_i *dominates* x'_i iff $p_i(x_i) > p_i(x'_i)$. Note that with this modified meaning of domination, the region of non-dominated strategies, X^*, differs as well.

The average payoff model has interesting properties, e.g., singleton profiles can be implemented for free.

Theorem 7. *If players maximize their average payoff, singleton strategy profiles are always 0-implementable if there are at least two players with at least two strategies.*

Theorem 7 implies that entire strategy profile regions O are 0-implementable as well: we just have to implement any singleton inside O.

Corollary 3. *In average strategy games where every player has at least two strategies, every strategy profile region can be implemented for free.*

Exact implementations can be implemented at no costs as well.

Theorem 8. *In average strategy games where $O_{-i} \subset X_{-i}\ \forall i \in N$, each strategy profile region has an exact 0-implementation.*

5.2 Risk-Averse Players

Instead of striving for a high payoff on average, the players might be cautious or *risk-averse*. To account for such behavior, we adapt our model by assuming that the players seek to minimize the risk on missing out on benefits. In order to achieve this objective, they select strategies where the minimum gain is not less than any other strategy's minimum gain. If there is more than one strategy with this property, the risk-averse player can choose a strategy among these, where the average of the benefits is maximal. More formally, let $min_i := \max_{x_i \in X_i}(\min_{x_{-i} \in X_{-i}}(U_i(x_i, x_{-i})))$ and $\varnothing_X f(x) := \frac{1}{|X|} \cdot \sum_{x \in X} f(x)$. Then Player i selects a strategy m satisfying $m = \arg\max_{m \in M}(\varnothing_{X_{-i}} U_i(m, x_{-i}))$, where $M = \{x_i | \forall x_{-i} \quad U_i(x_i, x_{-i}) = min_i\}$.

Theorem 9. *For risk-averse players the implementation cost of a singleton $z \in X$ is $k(z) = \sum_{i=1}^{n} \max(0, min_i - U_i(z))$*

For strategy profile regions, the situation with risk-averse players differs from the standard model considerably.

Theorem 10. *For risk-averse players the implementation cost for a strategy profile region $O \subset X$ is $k(O) = \min_{o \in O} \sum_{i=1}^{n} \max(0, min_i - U_i(o))$.*

In Section 3, we conjectured the problem of computing $k(O)$ to be $\mathbf{NP}$-complete for both general and exact implementations. This is not the case for risk-averse players, as the following theorem states.

Theorem 11. *$\mathcal{ALG}_{risk}$ computes $k(O)$ in time $\mathrm{O}(n|X|^2)$, thus the problem of computing k for risk-averse agents is in $\mathbf{P}$.*

Algorithm 5 Risk-averse Players: Exact Implementation

```
Input: Game $G$, target region $O$, $O_i \cap X_i^* = \emptyset \; \forall i \in N$
Output: $V$
 1: compute $X^*$;
 2: $V_i(z) = 0$ for all $i \in N, z \in X$;
 3: for all $i \in N$ do
 4:     $V_i(x_i, x_{-i})$:=$\infty \quad \forall x_i \in O_i$, $x_{-i} \in X_{-i} \setminus O_{-i}$;
 5:     $V_i(x_i, x_{-i}) := \max(0, min_i - U_i(x_i, x_{-i})) \quad \forall x_i \in O_i$,
        $x_{-i} \in X_{-i}$;
 6:     if $O_{-i} = X_{-i}$ then
 7:         if $\tau(O_i) > \tau(X_i^*)$ then
 8:             if $|X_i| + \epsilon|O_i| > |X_i| + \sum_{o_i} \delta(o_i)$ then
 9:                 $V_i(o_i, x_{-i})$:=$V_i(o_i, x_{-i}) + \delta(o_i) \quad \forall o_i, x_{-i}$;
10:             else
11:                 $V_i(o_i, x_{-i})$:=$V_i(o_i, x_{-i}) + \epsilon \quad \forall o_i, x_{-i}$;
12:         else
13:             if $\epsilon|O_i| > \sum_{o_i} [\epsilon + \delta(o_i)]$ then
14:                 $V_i(o_i, x_{-i})$:=$V_i(o_i, x_{-i}) + \epsilon + \delta(o_i) \quad \forall o_i, x_{-i}$;
15:             else
16:                 $V_i(o_i, x_{-i})$:=$V_i(o_i, x_{-i}) + \epsilon \quad \forall o_i, x_{-i}$;
17: return $V$;
```

5.3 Round-based Mechanisms

The previous sections dealt with static models only. Now, we extend our analysis to dynamic, round-based games, where the designer offers payments to the players after each round in order to make them change strategies. This opens many questions: For example, imagine a concrete game such as a *network creation game* [6] where all players are stuck in a costly Nash equilibrium. The goal of a mechanism designer could then be to guide the players into another, better Nash equilibrium. Many such extensions are reasonable; due to space constraints, only one model is presented in more detail.

In a dynamic game, we regard a strategy profile as a state in which the participants find themselves. In a network context, each $x \in X$ could represent one particular network topology. We presume to find the game in an initial starting state $s^{T=0} \in X$ and that, in state $s^{T=t}$, each Player i only sees the states she can reach by changing her strategy given the other players remain with their chosen strategies. Thus Player i

sees only strategy profiles in $X^{T=t}_{visible,i} = X_i \times \{s^{T=t}_{-i}\}$ in round t. In every round t, the mechanism designer offers the players a payment matrix $V^{T=t}$ (in addition to the game's static payoff matrix U). Then all players switch to their best visible strategy (which is any best response $B_i(s^{T=t}_{-i})$), and the game's state changes to $s^{T=t+1}$. Before the next round starts, the mechanism designer disburses the payments $V^{T=t}(s^{T=t+1})$ offered for the newly reached state. The same procedure is repeated until the mechanism designer decides to stop the game. We prove that a mechanism designer can guide the players to any strategy profile at zero costs in two rounds.

Theorem 12. *Starting in an arbitrary strategy profile, a dynamic mechanism can be designed to lead the players to any strategy profile without any expenses in at most two rounds if* $|X_i| \geq 3 \; \forall i \in N$.

6 Conclusion

Rendering distributed systems robust to non-cooperative behavior has become an important research topic. This paper has studied the fundamental question: Which outcomes can be implemented by promising players money while the eventual payments are bounded? We have presented algorithms for various objectives yielding implementations of low cost. Furthermore, it has been shown that a greedy algorithm performs well for random games. Finally, this paper has initiated the study of risk-averse players and round-based games.

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