Maximin-Aware Allocations of Indivisible Goods*

Extended Abstract

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ABSTRACT

We study envy-free allocations of indivisible goods to agents in settings where each agent is unaware of the bundles (or allocated goods) of other agents. In particular, we propose maximin aware (MMA) fairness measure, which guarantees that every agent, given the bundle allocated to her, is aware that she does not get the worst bundle, even if she does not know how the other goods are distributed. We also introduce two of its relaxations, MMA1 and MMAX. We show that MMA1 and MMAX potentially have stronger egalitarian guarantees than EF1 and are easier to achieve than MMS and EFX. Finally, we present a polynomial-time algorithm, which computes an allocation such that every agent is either $\frac{1}{2}$-approximate MMA or exactly MMAX. Interestingly, the returned allocation is also $\frac{1}{2}$-approximate EFX when all agents have subadditive valuations, which answers an open question left in [Plaut and Roughgarden, SODA 2018].

KEYWORDS

Fair allocation; maximin aware; envy-free

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1 INTRODUCTION

In the last few years or so, there has been a tremendous demand for fair division services to provide systematic and *fair* ways of dividing a set of indivisible m goods (denoted by M) such as tasks, courses, and properties among a group of n agents (denoted by N) so that the agents do not envy each other. To capture the fairness of an allocation, which is arguably initiated by the work of [4], *envy-freeness (EF)* is often used to ensure that each agent should not envy or prefer the allocated goods of other agents. Since an EF allocation barely exists[1], people study its relaxations, such as envy-freeness up to one good (EF1) and envy-freeness up to any good (EFX). The work of [2] introduced the relaxed concept of EF1, which requires that each agent's value for a bundle is at least as much as her value for every other agent's bundle minus a single good (in the bundle). It is shown in [6] that an EF1 allocation always exists, and can be found in polynomial time. [3] introduced the strictly stronger fairness notation than EF1, EFX, where the comparison is made to "any" single good instead of "a" single good. The state-of-the-art results show that an EFX allocation exists in the following settings: (1) there are 2 agents, or (2) there is any number of agents but all of them have the identical valuation [7]. It is still an open question whether an EFX allocation exists in general, even for additive valuations.

In this paper, we study an envy-free allocation domain where the planner of the division tasks wishes to withhold allocation information of others from the user or the user simply does not know the allocation of others in the system. There are a couple of good reasons why it is desirable for the planner to withhold such information. First, in many private fair allocations of goods such as tasks or gifts, the planner requires the system to preserve anonymity as not to give away the received bundles of other agents. Second, due to the large number of (unrelated) agents and items that could be potentially be involved in the division tasks (e.g., on the Internet such as MTurks), it is not meaningful for the planner to provide such information due to various reasons. Motivated by this domain, we focus on answering the following questions. *When indivisible goods are to be allocated among unaware agents, what is the appropriate envy-free notion and how efficiently can the allocation be found subject to the envy-free notion?*

Proportionality (PROP) and maximin share (MMS) [2] are two widely studied and well accepted fair allocation notions, both of which are defined for unaware agents. In PROP, it is required that the value of every agent's bundle is at least a $\frac{1}{n}$ fraction of her value for the whole goods, where n is the number of agents. It is well known that such an allocation barely exists for indivisible goods, thus a weaker and more realistic notion is desired for indivisible goods. MMS is a proper relaxation of PROP, which studies an adversarial situation: when the goods are partitioned into bundles and an agent would always get the least preferred bundle of goods, what is

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[1]For example, there are two agents but only one good.

the best way she can partition the goods. The value of such a bundle is the MMS value of the agent. In addition to its non-existence result, MMS allocation only guarantees each agent's best minimum value, and the value of some agent's bundle can still be the least compared with others, which may cause significant envy.

Recently, [1] introduced epistemic envy-free (EEF) notion to study unaware setting. With respect to EEF, each agent is satisfied if there exists one reallocation of the goods that she does not get among the other agents, such that her value for her bundle is at least as good as every bundle in this reallocation. This measure is not robust if the reallocations are restricted by agent's reasoning (e.g., adversarial settings in MMS). Moreover, it can be shown that EEF and PROP allocations (and their relaxations, e.g., removing any item) barely exist and cannot be properly approximated[2].

2 MAXIMIN-AWARE ALLOCATIONS

In this paper, we focus on modeling the envy-freeness for indivisible goods allocation as well as deriving new algorithms to find (approximately) fair allocations in an unaware environment.

We first define maximin-aware (MMA) allocations as follows.

Definition 2.1. An allocation A is called maximin-aware (MMA) if for any agent $i \in N$, $v_i(A_i) \geq \mathrm{MMS}_i(A_{-i}, n-1)$.

An MMA allocation guarantees that the agent's bundle value is at least as much as her value for some other agent's bundle, no matter how the remaining goods are distributed, i.e., there is always somebody who gets no more than her. MMA combines the notion of epistemic envy-freeness [1] and MMS [2] where each agent may not know or care about the exact allocation of the remaining goods to other agents, but can still guarantee that she does not envy everyone.

Throughout the whole paper, we call a valuation v (1) *additive* if $v(S) = \sum_{j \in S} v(j)$ for each $S \subseteq M$; (2) *binary additive* (BA) if v is additive, and $v(j) \in \{0, 1\}$ for any good j; (3) *submodular* (SM) if for any $S \subseteq T$ and $e \in M \setminus T$, $v(S \cup \{e\}) - v(S) \geq v(T \cup \{e\}) - v(T)$; (4) *subadditive* (SA) if $v(S \cup T) \leq v(S) + v(T)$ for any $S, T \subseteq M$. When we say $X \overset{type}{\Longrightarrow} Y$, we mean that every X allocation is also a Y allocation when agents have *type* valuations where X and Y are the fairness notions, e.g., MMA1 and MMS, and *type* is the function type, e.g. SA and SM.

Similar to MMS, MMA is a realistic definition for indivisible goods, in the sense that MMA is more approachable.

Lemma 2.2. *EF $\overset{A}{\Rightarrow}$ EEF $\overset{A}{\Rightarrow}$ PROP $\overset{A}{\Rightarrow}$ MMA ($\overset{BA}{\Longleftarrow}$ MMS).*

Actually, we show that all the implications in Lemma 2.2 are strict. However, MMA is still a strong requirement as such an allocation may not exist even when the agents have identical BA valuation. Thus we introduce two of its relaxations.

Definition 2.3 (Maximin-aware up to one good (MMA1)). For any $\alpha \in [0, 1]$, an allocation $A = (A_i)_{i \in N}$ is α-MMA1 if for any i,

$$v_i(A_i) \geq \alpha \cdot \mathrm{MMS}_i(A_{-i} \setminus \{e\}, n-1)$$

for some $e \in A_{-i}$. The allocation is MMA1 when $\alpha = 1$.

[2]Consider three agents with two goods such that every agent has value 1 for each good. Then the agent that receives no good has no bounded guarantee for both PROP and EEF (and their relaxations by removing any item).

Definition 2.4 (Maximin-aware up to any good (MMAX)). For any $\alpha \in [0, 1]$, an allocation $A = (A_i)_{i \in N}$ is α-MMAX if for any i,

$$v_i(A_i) \geq \alpha \cdot \mathrm{MMS}_i(A_{-i} \setminus \{e\}, n-1)$$

for any $e \in A_{-i}$. The allocation is MMAX when $\alpha = 1$.

We first note that while MMA1 (or MMAX) is a relaxed version of MMA, it potentially has better egalitarian guarantee than EF1 allocations. By definition, an MMA1 allocation A guarantees that for each agent i and her favorite item $e \in A_{-i}$ (suppose $e \in A_k$), $v_i(A_i)$ is at least as large as the worst bundle in *any* $(n-1)$-partition of $A_{-i} \setminus \{e\}$. However, if A is EF1, it means there *exists* an $(n-1)$-partition of $A_{-i} \setminus \{e\}$, i.e., $A_{i'}$ for $i' \neq k$ and $A_k \setminus \{e\}$, such that $v_i(A_i)$ is at least as large as the worst bundle, i.e. $A_k \setminus \{e\}$.

Next we compare MMA1/MMAX with existing fairness notions.

Lemma 2.5. *MMS $\overset{SM}{\Longrightarrow}$ MMA1 $\overset{BA}{\Longleftrightarrow}$ MMS and EFX $\overset{BA}{\Longrightarrow}$ MMAX.*

Again, we show that all the implications in Lemma 2.5 are actually strict. Since MMA1 is slightly weaker than MMS, it is expected that for a broader class of valuations, MMA1 allocations should be guaranteed to exist. Accordingly, we prove the following result.

Theorem 2.6. *In the following two cases an MMA1 allocation is guaranteed to exist: (1) there are at most three agents with submodular valuations or (2) any number of agents but all of them have identical submodular valuation.*

If the requirement of submodularity in Theorem 2.6 is replaced by strictly increasing subadditivity, an MMAX allocation is guaranteed to exist. In contrast, MMS allocation may not exist even for three agents with additive valuations [5] and an EFX allocation is only guaranteed to exist when there are two agents [7]. Thus, MMA1 and MMAX are good alternative criteria for the case when classic fairness cannot be guaranteed.

Finally, we present a polynomial-time algorithm to approximately compute a fair allocation.

Theorem 2.7. *There is an algorithm that computes an allocation in polynomial time which is (1) $\frac{1}{2}$-EFX and EF1 when all agents have subadditive valuations; (2) either $\frac{1}{2}$-MMA or exact MMAX to each agent when the valuations are additive.*

It is shown in [7] that a $\frac{1}{2}$-EFX allocation exists for general subadditive valuations, but finding it may need exponential time, which leaves an open question whether such an allocation can be found efficiently. Theorem 2.7 answers this question affirmatively.

3 CONCLUSION AND FUTURE DIRECTIONS

In this paper, we introduced novel fairness notions MMA and its two relaxations MMA1 and MMAX in an unaware environment. We study their connections with other fairness notations, and propose an efficient algorithm for computing allocations that is (approximately) MMA and MMAX.

We leave the general existence of MMA1 and MMAX allocations for a broader class of valuations as a future direction. Another promising direction would be to extend our work to other preference representations, including ordinal preferences, or to chores instead of goods. Finally, it will be very interesting to obtain analogues concepts for asymmetric agents, where all the agents have different entitlements or shares.

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