A Causal Semantics for the Edge Clique Cover Problem

Alex Markham and Moritz Grosse-Wentrup

Research Group Neuroinformatics, University of Vienna

contact: `alexander.markham@univie.ac.at`

website: `https://causal.dev/`

Abstract

We consider the task of causal structure learning over a set of measurement variables with no direct causal relations and whose dependencies are induced by unobserved latent variables. We call this the *measurement dependence inducing latent* (MeDIL) Causal Model, or MCM, framework. We show that this task can be framed in terms of the graph theoretical problem of finding edge clique covers, resulting in a simple algorithm for returning minimal MeDIL causal models (minMCMs). This algorithm is non-parametric, requiring no assumptions about linearity or Gaussianity. Furthermore, despite these rather weak and general assumptions, we are able to show that *minimality* in minMCMs implies three rather specific and interesting properties: first, minMCMs lower bound (i) the number of latent causal variables and (ii) the number of functional causal relations that are required to model a complex system at *any* level of granularity; second, a minMCM contains no causal links between the latent variables; and third, in contrast to factor analysis, a minMCM may require more latent than measurement variables.

The Edge Clique Cover (ECC) Problem

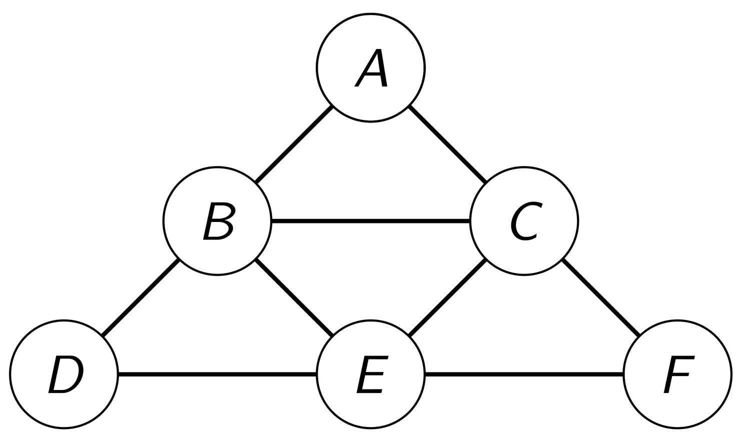

Figure 1: example undirected graph

The set of all maximal cliques for the above graph (i.e., the set consisting of all four 3-graphs) is an *edge clique cover* (ECC). Notice, however, that the clique formed by nodes A, B, and C can be removed and the three remaining cliques still cover all edeges. These three cliques together form a *minimal ECC*. Finding a minimal ECC is more complicated and has higher complexity than finding the set of all maximal cliques.

Furthermore, there are two diffirent kinds of minimality, clique-minimal and assignment-minimal:

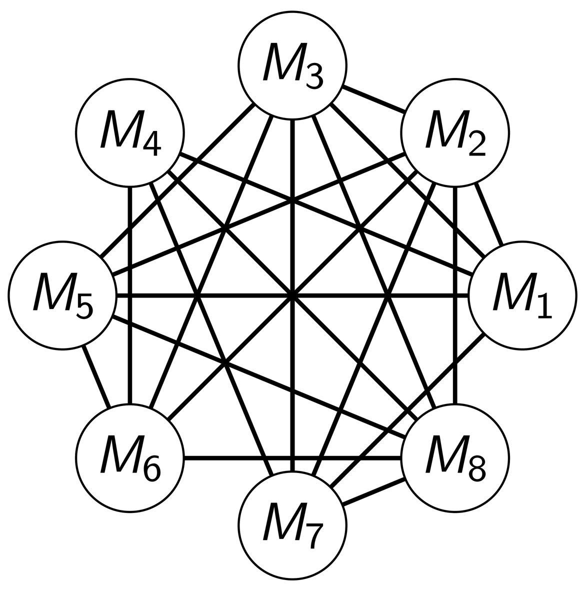

Figure 2(a): example undirected graph $D(\mathbf{M})$ over variables $\mathbf{M} = \{M_1, \dots, M_8\}$

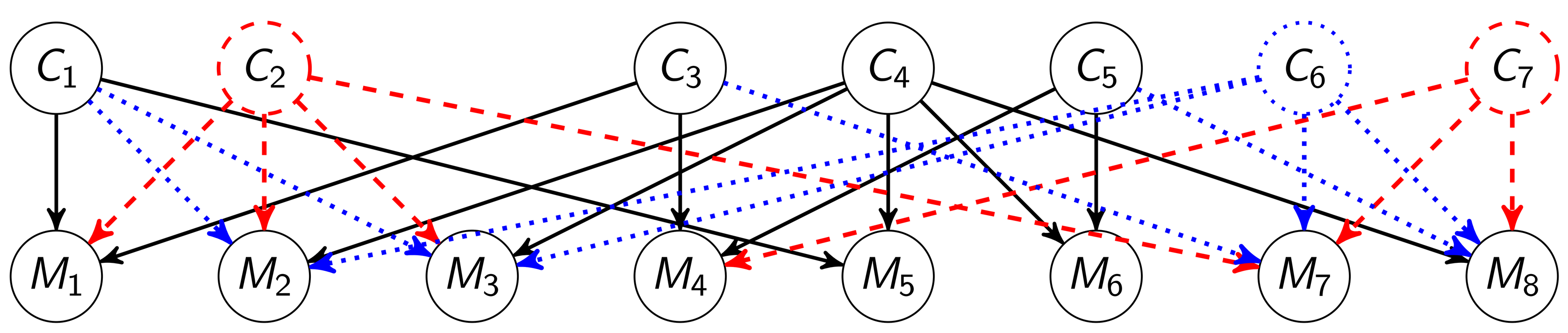

Figure 2(b): each C_i corresponds to a maximal clique in $D(\mathbf{M})$ and each directed edge represents assignment—dashed red edges/vertices are redundant for clique-minimality while blue dotted edges/vertices are redundant for assignment-minimality;

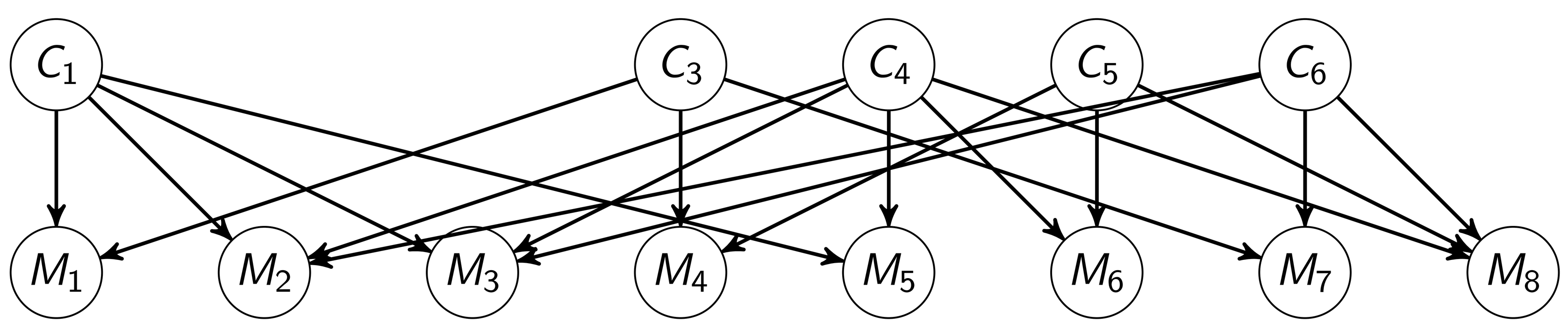

Figure 2(c): 5 cliques C_i and 19 assignments for clique-minimal ECC over $D(\mathbf{M})$

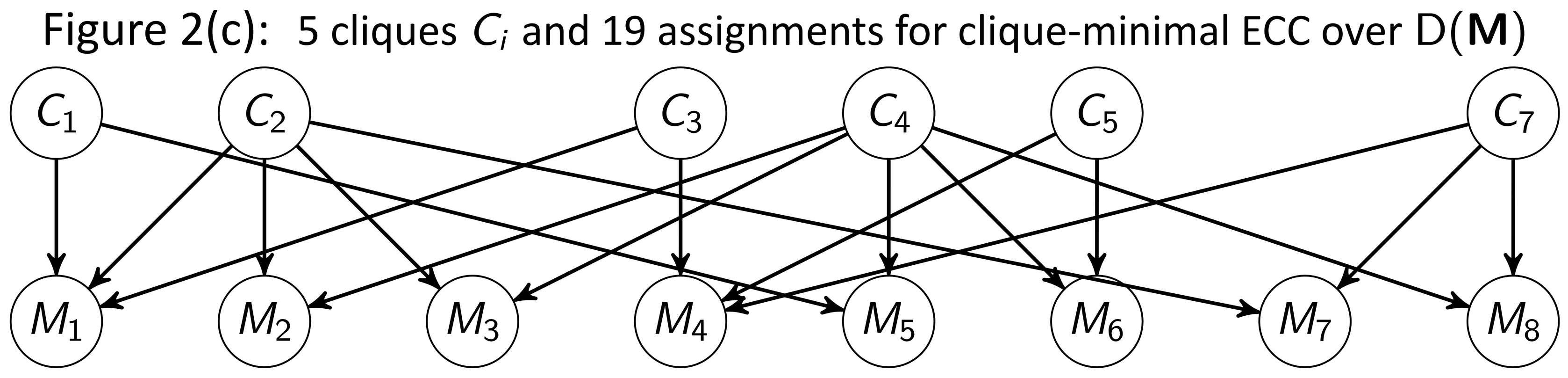

Figure 2(d): 6 cliques C_i and 18 assignments for assignment-minimal ECC over $D(\mathbf{M})$

MeDIL Causal Models

Definition 1 (Measurement Dependence Inducing Latent Causal Model (MCM)). A graphical MCM is a DAG, given by the triple $\mathcal{G} = \langle \mathbf{L}, \mathbf{M}, \mathbf{E} \rangle$. $\mathbf{L}$ and $\mathbf{M}$ are disjoint sets of vertices, while $\mathbf{E}$ is a set of directed edges between these vertices, subject to the following constraints:

1. all vertices in $\mathbf{M}$ have in-degree of at least 1 and out-degree of 0
2. all vertices in $\mathbf{L}$ have out-degree of at least 1
3. $\mathbf{E}$ contains no cycles

To learn a minMCM for a given distribution of measurement variables, we represent the measurement variables $\mathbf{M}$ as an *undirected dependency graph* upon which ECC finding algorithms can be applied. We denote this graph $D(\mathbf{M})$, and construct it by putting an undirected edge between two measurement variables if and only if they are unconditionally dependent. These dependencies can be learned from a set of samples via permutation-based hypothesis testing using non-linear measures of dependence, such as the Hilbert-Schmidt Independence Criterion (HSIC) or the distance correlation.

Algorithm 1: constructing a minimal MeDIL causal model (minMCM)

Input : $D(\mathbf{M})$ over the measurement variables $\mathbf{M}$
Output: vertex-minimal or assignment-minimal MCM $\mathcal{G}$ over $\mathbf{M}$

```
initialize edgeless graph with a vertex for each M ∈ M;
use find_cm or find_am to get an edge clique cover of D(M);
for each clique C in the cover do
    add vertex L with edges directed to each M ∈ C;
end
```

Some interesting properties include:

- minMCMs lower bound (i) the number of latent causal variables and (ii) the number of functional causal relations that are required to induce the measurement variables
- minMCM has property that for all measurement variables, conditional independence relations are implied by unconditional independence relations (this is related to the Global Markov Property of a Markov Random Field)
- a minMCM contains no causal links between the latent variables
- in contrast to factor analysis or independent component analysis, a minMCM may require more latent than measurement variables.

Conclusion

- some (in)dependence structures cannot be represented by a DAG over the corresponding variables but *can* be represented by an undirected dependency graph
- these structures are common for *measurement* variables, which are noisy copies or combinations of unobserved latents, e.g., in fMRI, calcium imaging, and psychiatric or econometric questionnaire data
- we propose the MeDIL causal model framework for use on such data and provide an algorithm for finding a minimal MCM for a given distribution of measurement variables.