

# A New Lower Bound for the Minimal Singular Value for Real Non-Singular Matrices by a Matrix Norm and Determinant

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## Abstract

A new lower bound on minimal singular values of real matrices based on Frobenius norm and determinant is presented. We show that under certain assumptions on matrix  $A$  this estimate is sharper than a recent bound from Hong and Pan based on a matrix norm and determinant.

**Keywords:** real non-singular matrix, singular values, lower bound

## 1 Introduction

The singular values or eigenvalues of real matrices are fundamental quantities describing the properties of a given matrix. They are however difficult to evaluate in general. It is useful to know at least approximate values of an interval of their occurrence. The first bounds for eigenvalues were obtained more than a hundred years ago. The first paper using traces in eigenvalue inequalities was from Schur in 1909 [5]. Possibly the best-known inequality on eigenvalues is from Gerschgorin in 1931 [3]. Recently, several other lower bounds have been proposed for the smallest singular value of a square matrix, such as Johnson's bound, Brauer-type bound, Li's bound and Ostrowski-type bound [6, 7, 1, 9, 12].

In our paper we deal with the interval bound on the minimal singular values derived by means of a matrix norm or determinant by applying a stronger version of the so-called Kantorovich inequality from Diaz and Metcalf [2].

## 2 Preliminary Notes

Let  $A$  be a  $n \times n$ ,  $n \geq 2$  matrix with real elements. Let  $\|A\|_E = (\sum_{i,j=1}^n |a_{ij}|^2)^{1/2}$  be the Frobenius norm of matrix  $A$ . Trace of a  $n \times n$  matrix  $A$  denotes  $tr(A) = \sum_{i=1}^n a_{ii}$ . The spectral norm of the matrix  $A$  is  $\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i}$ , where  $\lambda_i$  is eigenvalue of  $A^T A$ . If  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of the matrix  $A$ , then  $\det A = \lambda_1 \lambda_2 \dots \lambda_n$ . Denote the smallest singular value of  $A$  by  $\sigma_n$  and its largest singular value by  $\sigma_1$ . It holds that  $\|A\|_E^2 = \sum_{i=1}^n \sigma_i^2 = tr(A^T A)$ , where trace  $tr(A^T A) = \sum_{i=1}^n \sigma_i^2$ .

Hong and Pan gave in [4] a lower bound for  $\sigma_n$  for a nonsingular matrix as

$$\sigma_{min} > \left(\frac{n-1}{n}\right)^{(n-1)/2} |\det A|. \quad (1)$$

In 2007 Turkmen and Civcic in [10] also used matrix norm and determinant for finding upper bounds for maximal and minimal singular value of positive definite matrices.

For symmetric positive definite matrix  $A$ , one can suppose  $\|A\|_2 = 1$ , i.e. that the matrix  $A$  is normalized, where  $\|\cdot\|_2$  is the spectral norm. Consequently for its condition number is  $\kappa(A) = \|A\|_2 \|A^{-1}\|_2 = \frac{1}{\sigma_n}$ . The matrix normalization can be always achieved by multiplying the set of equations  $Ax = b$  by a suitable constant or for example by the divisive normalization defined by Weiss [11] or Ng et al. [8], which uses the Laplacian  $L$  of the symmetric positive definite matrix  $A$ . The transformation is defined by  $D^{-1/2} A D^{-1/2}$ , where  $D = \{d_{ij}\}_{i,j=1}^n$  and  $d_{ij} = 0$  for  $i \neq j$  and  $d_{ij} = \sum_{j=1}^n a_{ij}$  for  $i = j$ , where  $A = \{a_{ij}\}_{i,j=1}^n$ .

## 3 Main Results

**Theorem 3.1** *Let  $A$  be a nonsingular matrix with singular values  $\sigma_i$  so that  $\sigma_{max} = \sigma_1 \geq \dots \geq \sigma_n = \sigma_{min}$  and let  $\sigma_{max} \neq \sigma_{min}$ . Let  $\|A\|_E = (\sum_{i,j=1}^n |a_{ij}|^2)^{1/2}$  be the Frobenius norm of matrix  $A$ .*

(i) *Then for its minimal and maximal singular values holds*

$$0 < \left( \frac{\|A\|_E^2 - n\sigma_{max}^2}{n(1 - \frac{\sigma_{max}^2}{|\det A|^{2/n}})} \right)^{1/2} < \sigma_{min}. \quad (2)$$

(ii) *For  $\sigma_{max} = 1$  (supposing that  $|\det A| \neq 1$ ) holds*

$$0 < \left( \frac{|\det A|^{2/n} (\|A\|^2 - n)}{n(|\det A|^{2/n} - 1)} \right)^{1/2} < \sigma_{min}.$$

Proof:

(i) We will apply the following result of Diaz and Metcalf [2] which is a stronger form of Pólya-Szegő and Kantorovich’s inequality. Let the real numbers  $a_k \neq 0$  and  $b_k$  ( $k = 1, \dots, n$ ) satisfy  $m \leq \frac{b_k}{a_k} \leq M$ . Then  $\sum_{k=1}^m b_k^2 + mM \sum_{k=1}^n a_k^2 \leq (m + M) \sum_{k=1}^n a_k b_k$ .

Let  $b_k = \sigma_k$ ,  $a_k = \frac{1}{\sigma_k}$ ,  $m = \sigma_{min}^2$ ,  $M = \sigma_{max}^2$ , let  $m \neq M$ . Then from the Diaz and Metcalf’s inequality follows, that  $\sum \sigma_k^2 + mM \sum \frac{1}{\sigma_k^2} \leq (M + m)n$ . From the latter inequality and from the relationship of arithmetic and geometric mean follows  $\|A\|_E^2 < Mn + mn - \frac{mMn}{|detA|^{2/n}}$  and also  $\frac{\|A\|_E^2}{n} - M < m(1 - \frac{M}{|detA|^{2/n}})$  and from that follows  $\frac{\|A\|_E^2 - Mn}{n(1 - \frac{M}{|detA|^{2/n}})} < m = \sigma_{min}^2$  and the statement of the theorem follows.

It holds that  $0 < \frac{\|A\|_E^2 - Mn}{n(1 - \frac{M}{|detA|^{2/n}})}$ , since to be true must hold ( $\|A\|_E^2 - Mn > 0$  and  $1 - \frac{M}{|detA|^{2/n}} > 0$ ) or ( $\|A\|_E^2 - Mn < 0$  and  $1 - \frac{M}{|detA|^{2/n}} < 0$ ). The second case  $\|A\|_E^2 - Mn < 0$  and  $1 - \frac{M}{|detA|^{2/n}} < 0$  holds, since  $\|A\|_E^2 = \sum_{i=1}^n \sigma_i^2 < \sigma_{max}^2 n$  and  $\prod_{i=1}^n \sigma_i^2 = |detA|^2 < M^n = (\sigma_{max}^2)^n$ . (ii) follows from (i).

## 4 Comparison with the estimate from Hong and Pan

We will show that our estimate of minimal singular value is under certain conditions sharper than the estimate from from Hong and Pan [4].

**Theorem 4.1** *Let for a  $n \times n$ - nonsingular matrix  $A$  with  $n > 1$  hold for its singular values  $\sigma_i$   $\sigma_{max} = \sigma_1 = 1 \geq \dots \geq \sigma_n = \sigma_{min}$  and let  $\sigma_{max} \neq \sigma_{min}$ . If also  $0 < |detA|, \|A\|_E < 1$  then*

$$\left(\frac{n-1}{n}\right)^{(n-1)/2} |detA| < \left(\frac{|detA|^{2/n} (\|A\|^2 - n)}{n(|detA|^{2/n} - 1)}\right)^{1/2} \tag{3}$$

*i.e. our estimate is sharper than the estimate (1) from Hong and Pan.*

Proof:

Since  $0 < |detA|, \|A\|_E < 1$ , is (3) equivalent to

$$\left(\frac{n-1}{n}\right)^{(n-1)} |detA|^2 < \frac{|detA|^{2/n} (\|A\|^2 - n)}{n(|detA|^{2/n} - 1)}. \tag{4}$$

Since between arithmetical mean  $A(\sigma_i^2)$  and geometrical mean  $G(\sigma_i^2)$  of  $\sigma_1^2, \dots, \sigma_n^2$  holds  $A(\sigma_i^2) \geq G(\sigma_i^2)$ , then is  $\frac{(\frac{1}{n}\|A\|_E^2 - 1)}{(|detA|^{2/n} - 1)} \geq 1$ . Assume by contradiction that

the opposite of (4) holds:

$$\frac{(\frac{1}{n}\|A\|_E^2 - 1)}{(|\det A|^{2/n} - 1)} \leq \left(\frac{n-1}{n}\right)^{(n-1)} |\det A|^{2-\frac{2}{n}}. \quad (5)$$

Since the right hand side of (5) is smaller than 1 then (5) cannot hold and brings (5) to a contradiction.

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