

Three Bounds on the Minimal Singular Value: A Comparison

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Abstract

We presented a new lower bound on minimal singular values of real matrices based on Frobenius norm and determinant and showed in [4] that under certain assumptions on the matrix is our estimate sharper than a recent lower bound from Hong and Pan [3]. In this paper we show, under which conditions is our lower bound sharper than two other recent lower bounds for minimal singular values based on a matrix norm and determinant, namely the bound from Piazza and Politi and the bound from Hou-Biao Li et al.

Keywords: real non-singular matrix, singular values, lower bound

1 Introduction

The singular values or eigenvalues of real matrices are fundamental quantities in matrix analysis and other mathematical fields. The term singular value relates to the distance between a matrix and the set of singular matrices. The singular values are difficult to evaluate in general. In some cases at least the knowledge of their approximate values is valuable. The first bounds for eigenvalues were obtained more than a hundred years ago. The first paper using traces in eigenvalue inequalities was from Schur in 1909 [9]. Possibly the best-known inequality on eigenvalues is from Gerschgorin in 1931 [2]. Recently, several other lower bounds have been proposed for the smallest singular value of a square matrix, such as Johnson's bound, Brauer-type bound, Li's bound and Ostrowski-type bound [6, 7, 1, 8, 13].

In this paper we deal only with the bounds on the minimal singular values using a matrix norm or determinant. We compare our bound from [4] with two other known bounds using a matrix norm or determinant.

2 Preliminary Notes

Let A be a $n \times n$, $n \geq 2$ matrix with real elements. Let $\|A\|_E = (\sum_{i,j=1}^n |a_{ij}|^2)^{1/2}$ be the Frobenius norm of matrix A . Trace of a $n \times n$ matrix A denotes $tr(A) = \sum_{i=1}^n a_{ii}$. The spectral norm of the matrix A is $\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i}$, where λ_i is eigenvalue of $A^T A$. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the matrix A , then $\det A = \lambda_1 \lambda_2 \dots \lambda_n$. Denote the smallest singular value of A by σ_n and its largest singular value by σ_1 . It holds that $\|A\|_E^2 = \sum_{i=1}^n \sigma_i^2 = tr(A^T A)$, where trace $tr(A^T A) = \sum_{i=1}^n \sigma_i^2$.

For symmetric positive definite matrix A , one can suppose $\|A\|_2 = 1$, i.e. that the matrix A is normalized and $\sigma_{max} = 1$. The matrix normalization can be always achieved by multiplying the set of equations $Ax = b$ by a suitable constant or for example by the divisive normalization defined by Weiss [12] or Ng et al. [10].

G. Piazza and T. Politi gave in 2002 [11] a lower bound on the minimal singular matrix with positive singular values as

$$\sigma_{min} \geq \frac{|\det A|}{2^{(n-2)/2} \|A\|_E}. \quad (1)$$

Hou-Biao Li et al. gave in 2010 in [5] another lower bound on minimal singular bound

$$\sigma_{min} \geq |\det A| \left(\frac{\sqrt{n-1}}{\|A\|_E} \right)^{n-1}. \quad (2)$$

These two lower bounds on the minimal singular value based on determinant and matrix norm will be compared with our bound given in the next section.

3 Main Results

3.1 Our lower bound

In [4] we proved the following theorem.

Theorem 3.1 *Let A be a nonsingular matrix with singular values σ_i so that $\sigma_{max} = \sigma_1 \geq \dots \geq \sigma_n = \sigma_{min}$ and let $\sigma_{max} \neq \sigma_{min}$. Let $\|A\|_E = (\sum_{i,j=1}^n |a_{ij}|^2)^{1/2}$ be the Frobenius norm of matrix A .*

(i) *Then for its minimal and maximal singular values holds*

$$0 < \left(\frac{\|A\|_E^2 - n\sigma_{max}^2}{n(1 - \frac{\sigma_{max}^2}{|\det A|^{2/n}})} \right)^{1/2} < \sigma_{min}. \quad (3)$$

(ii) For $\sigma_{max} = 1$ (supposing that $|\det A| \neq 1$) holds

$$0 < \left(\frac{|\det A|^{2/n}(\|A\|^2 - n)}{n(|\det A|^{2/n} - 1)} \right)^{1/2} < \sigma_{min}.$$

3.2 Comparison with the lower bound from Piazza and Politi

We will show that our estimate of minimal singular value is under certain conditions sharper than the estimate from from Piazza and Politi.

Theorem 3.2 *Let for a $n \times n$ - nonsingular matrix A with $n > 1$ hold for its singular values σ_i , $\sigma_{max} = \sigma_1 = 1 \geq \dots \geq \sigma_n = \sigma_{min}$ and let $\sigma_{max} \neq \sigma_{min}$ and $\|A\|_E \neq 0$. If for matrix A holds $|\det A| < (n2^{n-2})^{\frac{n}{2(n-2)}}$ then*

$$\frac{|\det A|}{2^{(n-2)/2}\|A\|_E} < \left(\frac{\|A\|_E^2 - n\sigma_{max}^2}{n(1 - \frac{\sigma_{max}^2}{|\det A|^{2/n}})} \right)^{1/2} \tag{4}$$

i.e. our estimate is sharper than the estimate from Piazza and Politi.

Proof:

Formula (4) in power 2 is

$$\frac{|\det A|^2}{2^{(n-2)}\|A\|_E^2} < \left(\frac{\|A\|_E^2 - n\sigma_{max}^2}{n(1 - \frac{\sigma_{max}^2}{|\det A|^{2/n}})} \right). \tag{5}$$

Since between arithmetical mean $A(\sigma_i^2)$ and geometrical mean $G(\sigma_i^2)$ of $\sigma_1^2, \dots, \sigma_n^2$ holds $A(\sigma_i^2) \geq G(\sigma_i^2)$ then is $X = \frac{(\frac{1}{n}\|A\|_E^2 - \sigma_{max}^2)}{(|\det A|^{2/n} - \sigma_{max}^2)} \geq 1$. The right side of (5) is $|\det A|^{2/n} \cdot \frac{(\frac{1}{n}\|A\|_E^2 - \sigma_{max}^2)}{(|\det A|^{2/n} - \sigma_{max}^2)}$. It will then suffice to prove that

$$\frac{|\det A|^{2 - \frac{2}{n}}}{\|A\|^2 2^{n-2}} < 1. \tag{6}$$

Since $A(\sigma_i^2) \geq G(\sigma_i^2)$ then also $|\det A|^{2/n} \leq \frac{1}{n}\|A\|^2$ and

$$\frac{1}{\|A\|^2} \leq \frac{1}{n|\det A|^{2/n}}. \tag{7}$$

The left side of (6) is $\frac{|\det A|^2}{\|A\|^2 |\det A|^{2/n} 2^{n-2}}$, which using (7) is smaller or equal to $\frac{|\det A|^2}{n|\det A|^{4/n} 2^{n-2}}$. The last value is smaller than 1 because of the condition on $\det A$ from the statement of the theorem.

3.3 Comparison with the lower bound from Hou-Biao Li et al.

We will show that under certain conditions is our lower bound of minimal singular value sharper than the lower bound from Hou-Biao Li et al.

Theorem 3.3 *Let for a $n \times n$ - nonsingular matrix A with $n > 1$ hold for its singular values σ_i , $\sigma_{max} = \sigma_1 = 1 \geq \dots \geq \sigma_n = \sigma_{min}$ and let $\sigma_{max} \neq \sigma_{min}$ and $\|A\|_E \neq 0$. If between $|\det A|$, $\|A\|_E$ holds $(n-1)^{\frac{1}{2}}|\det A|^{\frac{1}{n}} < \|A\|_E$ then*

$$\left(\frac{n-1}{\|A\|_E^2}\right)^{(n-1)/2}|\det A| < \left(\frac{\|A\|_E^2 - n\sigma_{max}^2}{n(1 - \frac{\sigma_{max}^2}{|\det A|^{2/n}})}\right)^{1/2} \quad (8)$$

i.e. our estimate is sharper than the estimate from Hou-Biao Li et al.

Proof:

Denote $P = X \cdot |\det A|^{2/n} = \frac{(\frac{1}{n}\|A\|_E^2 - \sigma_{max}^2)}{(|\det A|^{2/n} - \sigma_{max}^2)}|\det A|^{2/n}$, $L = \left(\frac{n-1}{\|A\|_E^2}\right)^{(n-1)}|\det A|^2$.

From the proof of the previous theorem we know that $X = \frac{(\frac{1}{n}\|A\|_E^2 - \sigma_{max}^2)}{(|\det A|^{2/n} - \sigma_{max}^2)} \geq 1$.

One is to prove that under the given conditions for $\|A\|_E$ and $|\det A|$ is $L < P$.

It is obvious that

$$\frac{1}{n} = \frac{1}{n-1}\left(1 - \frac{1}{n}\right). \quad (9)$$

From the condition $(n-1)^{\frac{1}{2}}|\det A|^{\frac{1}{n}} < \|A\|_E$ follows

$(n-1)|\det A|^{\frac{2}{n}} < \|A\|_E^2$ which is under the rewriting of $\frac{2}{n}$ by means of (9) equivalent to

$$(n-1)(|\det A|^{2-\frac{2}{n}})^{\frac{1}{n-1}} < \|A\|_E^2. \quad (10)$$

The $(n-1)$ -th power of (10) gives

$$(n-1)^{n-1}(|\det A|^{2-\frac{2}{n}}) < \|A\|_E^{2(n-1)}, \quad (11)$$

from which is

$$L = \left(\frac{(n-1)}{\|A\|_E^2}\right)^{n-1}|\det A|^2 < |\det A|^{2/n}. \quad (12)$$

Since $X \geq 1$, is also $L < |\det A|^{2/n} \cdot X = P$ which was to be proven.

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