# Covering Rectilinear Polygons with Area-Weighted Rectangles 

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#### Abstract

Representing a polygon using a set of simple shapes has numerous applications in different use-case scenarios. We consider the problem of covering the interior of a rectilinear polygon with holes by a set of area-weighted, axis-aligned rectangles such that the total weight of the rectangles in the cover is minimized. Already the unit-weight case is known to be $\mathcal{N} \mathcal{P}$-hard and the general problem has, to the best of our knowledge, not been studied experimentally before. We show a new basic property of optimal solutions of the weighted problem. This allows us to speed up existing algorithms for the unit-weight case, obtain an improved ILP formulation for both the weighted and unweighted problem, and develop several approximation algorithms and heuristics for the weighted case. All our algorithms are evaluated in a large experimental study on 186837 polygons combined with six cost functions, which provides evidence that our algorithms are both fast and yield close-to-optimal solutions in practice.


## Funding

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement No. 101019564 "The Design of Modern Fully Dynamic Data
 Structures (MoDynStruct)") and from the Austrian Science Fund (FWF) project Z 422-N, and project "Fast Algorithms for a Reactive Network Layer (ReactNet)", P 33775-N, with additional funding from the netidee SCIENCE Stiftung, 2020-2024.

## 1 Introduction

Representing a polygon's interior using simpler shapes is a relevant problem in many fields, including integrated circuit design [17], image compression [22] and image construction [29]. When using rectangles, the main goal is usually to minimize the number of rectangles used to represent the polygon. However, in some applications, there may be a cost associated with both the number of shapes used as well as their area. If shapes may overlap, this can lead to a solution with more shapes costing less than a solution with fewer shapes.

One such example are 2D video games in which sets of individual "tiles" may be more compactly represented as "objects". Each object requires a certain amount of time to initialize itself, after which each of its tiles requires a certain amount of time to be rendered. Depending on how long both actions take, it may be faster to have more objects, consisting of fewer total tiles, or fewer objects, consisting of more total tiles. Note that for the purpose of this comparison, if two or more tiles overlap, all of them are still rendered and thus contribute to the total runtime.

This motivates our study of the following Weighted Rectangle Cover problem (WRC): Given a rectilinear polygon, which may contain holes, together with a cost function, find a set of rectangles which covers the polygon's interior with minimum total cost. The problem generalizes the unit-weight case, which is known to be $\mathcal{N} \mathcal{P}$-hard $[26,8]$. Although results on approximability and experimental studies for the unit-weight case and the closely related rectilinear picture compression problem exist [3, 4, 18, 22], little is known about the weighted case.

In this work, we initiate the study of the weighted rectangle cover problem. Motivated by the applications above, we focus on cost functions that can be expressed as a linear function of the rectangle's area ('area cost') plus a constant ('creation cost'), but many of our results can be transferred to other cost functions.

## Contributions

- We introduce the concept of base rectangles as an alternative approach to (Hanan) grid rectangles and show that they can reduce the complexity of the discretized problem significantly.
- We prove that there always exists an optimal solution that is built from base rectangles.
- We give the first polynomially-sized ILP formulation for the WRC problem, which also improves the ILP formulations for the unit-weight case, due to base rectangles.
- We develop and analyze several algorithms to solve the WRC problem quickly that yield, in practice, a high solution quality. In particular, we introduce four weight-aware postprocessors, which can also be used to adapt algorithms for the unweighted case.
- We report on a large experimental evaluation of ten algorithms on 186837 nontrivial polygons in combination with six cost functions, which shows that our new algorithms are not only faster than the greedy weighted set cover algorithm, but also produce close-to-optimal solutions in practice.
Paper Outline. Section 2 and Section 3 start with related work and preliminaries. In Section 4, we introduce Base Rectangles. We describe our algorithms and postprocessors in Section 5, their experimental evaluation in Section 6, and conclude in Section 7.


Figure 1: Three optimal covers $C_{1}$ (left), $C_{2}$ (center), $C_{3}$ (right) for the same polygon with different parameters.
For $C_{1}, \beta=0,\left|C_{1}\right|=9$, and $\sum_{R \in C_{1}} \mathcal{A}(R)=376$.
For $C_{2}, \frac{\beta}{\alpha}=\frac{1}{3},\left|C_{2}\right|=13$, and $\sum_{R \in C_{2}} \mathcal{A}(R)=156$.
For $C_{3}, \frac{\beta}{\alpha}=2,\left|C_{3}\right|=15$, and $\sum_{R \in C_{3}} \mathcal{A}(R)=152$.

## 2 Related Work on Rectilinear Polygons

Unless denoted otherwise, we assume in the following that the polygon under consideration is rectilinear and all rectangles are axis-aligned.

Weighted Set Cover. Given $k$ subsets $\mathcal{S}=\left(S_{1}, S_{2}, \ldots, S_{k}\right)$ of a universe $U$, each of which has an associated weight $w\left(S_{j}\right)$, this problem asks to pick a subset $\mathcal{S}^{\prime} \subseteq \mathcal{S}$ whose union equals $U$ and minimizes $\sum_{S \in \mathcal{S}^{\prime}} w(S)$. Simple greedy $O(\log |U|)$-approximation algorithms exist both for the unweighted [25, 20] and for the weighted case [7]. Feige [11] showed that this approximation ratio is tight unless $\mathcal{P}=\mathcal{N} \mathcal{P}$. Weighted geometric set cover refers to a family of special cases, where the subsets $S_{j}$ are geometric objects [31]. Even et al. [10] give a randomized $O(\log O P T)$-approximation algorithm for the special case of geometric objects with so-called "low VC-dimension" with all weights greater than 1. Their algorithm is based on solving the relaxed linear program of the problem instance and then rounding it by probabilistically finding an $\varepsilon$-net of small size. As OPT can be arbitrarily larger than $|U|[1]$, this is not necessarily an improvement over the simple greedy algorithm. Varadarajan [31] gives a similar probabilistic algorithm which uses a "quasi-uniform" sampling of $\varepsilon$-nets to round the linear program solution and has an approximation ratio of $2^{O\left(\log ^{*} k\right)} \log k$, with $k=|\mathcal{S}|$. This improves over previous results if the union complexity, i.e., the complexity of the boundary of the union of all geometric objects, is near-linear. Note that the union complexity is in $O\left(r^{2}\right)$ for $r$ rectangles [21].

There are also interesting recent results for the unweighted case [2, 5], which are likewise centered around the computation of $\varepsilon$-nets and largely theoretical, though an efficient computation of $\varepsilon$-nets for disks has been implemented [5].

Rectilinear Polygon Partition. A partition (also called decomposition or dissection) of a rectilinear polygon into a minimum set of non-overlapping rectangles can, in contrast to the overlapping case, be computed in polynomial time even if the polygon contains holes. For a polygon with $n$ vertices, Ohtsuki [28] gives an algorithm with running time $O\left(n^{2.5}\right)$, which was later improved to $O\left(n^{1.5} \log n\right)$ [19]. The algorithm was in fact discovered several times $[24,12,9]$ and is described in more detail in Section 5.4.

A related problem, where instead of minimizing the number of rectangles, the goal is to minimize the total length of all segments used to create the partition is studied by

Lingas et al. [23]. They show that the problem can be solved in time $O\left(n^{4}\right)$ for arbitrary rectilinear hole-free polygons with $n$ vertices, and in $O\left(n^{3}\right)$ for "histogram" polygons. In the presence of holes, the problem becomes $\mathcal{N} \mathcal{P}$-hard.

Unweighted Rectangle Cover. The problem of covering a rectilinear polygon with holes by axis-aligned rectangles was proven to be $\mathcal{N} \mathcal{P}$-hard by Masek [26] in 1978, and remains $\mathcal{N} \mathcal{P}$-hard for hole-free polygons [8].

Assume that the input polygon has $n$ vertices (including the vertices of holes). Since the unweighted problem is known to be a special case of the set cover problem with a universe of size polynomial in $n$, it can be approximated likewise within an $O(\log n)$ factor. Furthermore, a polynomial-time $O(\sqrt{\log n})$-approximation algorithm exists, which is due to Kumar and Ramesh [3]. In Section 5.3, we will give a modified version of this algorithm and show that it can be implemented with running time $O\left(n^{2}\right)$. Berman and DasGupta [4] showed that the rectangle cover problem does not admit a polynomial-time approximation scheme unless $\mathcal{P}=\mathcal{N} \mathcal{P}$.

Improved results exist for certain special cases: For polygons that are vertically convex, Franzblau and Kleitman [14] give a $O\left(n^{2}\right)$-time algorithm that computes an optimal solution. A polygon is vertically (horizontally) convex if the straight-line segment connecting any two interior points that have the same $x$-coordinate ( $y$-coordinate) lies in the interior of the polygon. In particular, this implies that the polygon is hole-free. Franzblau [13] showed that for a polygon with $h$ holes, a partition has at most $2 \cdot O P T+h-1$ rectangles, thus giving a 2-approximation algorithm for hole-free polygons. She also gives a $O(n \log n)$-time algorithm that produces a rectangle cover of size $O(O P T \cdot \log O P T)$. The problem was also studied in [27].

Rectilinear Picture Compression. Given a binary matrix, the rectilinear picture compression problem asks to represent it as a minimum set of rectangular submatrices containing only one-entries, such that every one-entry is contained in at least one of our submatrices. This is equivalent to an unweighted and integral version of the rectangle cover problem, where in addition, the size of the input equals the area of the bounding box of the represented polygon, i.e., $w \cdot h$ if $w$ and $h$ denote the width and height of the input polygon's bounding box. Note that $w$ and $h$ are not bounded by the number of vertices of the polygon in general.

Heinrich-Litan and Lübbecke [18] give primal and dual integer linear programming formulations for this problem, further ones are discussed by Koch and Marenco [22]. An algorithm that produces good results in practice is presented by the same authors [18]. It is based on the greedy set cover algorithm (cf. Section 5.2), but extends it by picking certain rectangles which are guaranteed to be part of some optimal cover before invoking the greedy set cover algorithm to cover the remaining parts of the polygon, pruning any redundant rectangles in a postprocessing step. This algorithm is based on an earlier one [16], which has worst-case runtime complexity $O\left(\max \{w, h\}^{5}\right)$.

Fast approaches for large instances with potentially improved quality are discussed by Koch and Marenco [22]. Their approaches work by finding an initial cover for the polygon by looking for large rectangles, grouping the pixels of the polygon into so-called atomic rectangles using this initial cover and then choosing a subset of the initial cover to cover the obtained atomic rectangles.

To the best of our knowledge, no specific algorithms or results for the weighted rectangle cover problem exist.


Figure 2: Concave and convex vertex w.r.t. the interior of the polygon.

## 3 Preliminaries

We consider a non-self-intersecting, rectilinear polygon $P=(E, \mathcal{H})$, which is specified by a cyclic, alternating list of horizontal and vertical edges $E$ along with a set of holes $\mathcal{H}$, where each hole $H \in \mathcal{H}$ is itself a hole-free, non-self-intersecting, rectilinear polygon that is strictly contained in $P$. A polygon is said to be hole-free if $\mathcal{H}=\emptyset$. An edge $e=\overline{u v}$ is either a horizontal or vertical segment connecting the two vertices $u$ and $v$, where $u, v \in \mathbb{R}_{\geq 0}^{2}$. A vertex $v$ is integral if $v \in \mathbb{N}^{2}$. For every pair of subsequent edges $e_{1}=\overline{u_{1} v_{1}}$ and $\bar{e}_{2}=\overline{u_{2} v_{2}}$ from $E$, if $v_{1}=u_{2}$ then $e_{1}$ and $e_{2}$ are called adjacent. Thus, $E$ is fully determined by either the subsequence of horizontal or the subsequence of vertical edges. A pair of non-adjacent edges $e_{1}, e_{2}$ is said to be intersecting if the intersection of their segments is nonempty. A polygon is self-intersecting if it has a pair of intersecting edges.

In the following, we assume that all polygons are rectilinear, non-self-intersecting, and that two holes of the same polygon may only intersect in a single common vertex ${ }^{1}$.

The interior $\mathcal{I}(P)$ of a polygon $P=(E, \mathcal{H})$ is the bounded region enclosed by $E$ minus the interiors of all holes in $\mathcal{H}$. A polygon $C$ is said to be (strictly) contained in a polygon $P$ if $\mathcal{I}(C)$ is a (strict) subset of $\mathcal{I}(P)$. The area $\mathcal{A}(P)$ is the area of $\mathcal{I}(P)$ in $\mathbb{R}^{2}$. A point $x \in \mathcal{I}(P)$ is an inner point if $x$ is not part of the boundary of neither $P$ nor any of its holes. A vertex of a polygon is called convex if the interior angle between its two edges is $\frac{\pi}{2}$ and concave otherwise. Note that the definition of convex and concave depends on the interior, i.e., a convex vertex of a polygon $H$ may be concave w.r.t. another polygon $P$ if $H$ is added as a hole to $P$, see also Figure 2.

For $P=(E, \mathcal{H})$, we use $\mathcal{E}(P)$ and $\mathcal{V}(P)$ to refer to the union of all edges and vertices in $E$ and $\mathcal{H}$, respectively, i.e., $\mathcal{E}(P)=E \cup \bigcup_{H \in \mathcal{H}} \mathcal{E}(H)$, and $\mathcal{V}(P)$ is the set of all vertices in $\mathcal{E}(P)$. We let $n(P)$ denote the size of $P$, where $n(P):=\frac{1}{2}|\mathcal{E}(P)|$, i.e., the number of horizontal (or vertical) edges in $\mathcal{E}(P)$. For brevity, we just use $n$ if no ambiguity arises. Note that $|\mathcal{V}(P)| \leq 2 n$ due to vertex-intersecting holes.

A rectangle $R$ is a hole-free polygon with $n(R)=2$. For simplicity, we represent $R$ as a pair of vertices $\left(x_{1}, x_{2}\right)$, where $x_{1}=: \Gamma(R)$ and $\left.x_{2}=:\right\lrcorner(R)$ denote the top-left and bottom-right vertex of $R$, respectively. The bounding box $\mathbb{B}(P)$ of a polygon $P$ is the rectangle $R$ such that $\mathcal{I}(P) \subseteq \mathcal{I}(R)$ and $\mathcal{A}(R)$ is minimal. A pixel is a unit-area, square rectangle with only integral vertices. A rectangle $M$ is said to be a maximal rectangle of $P$ if $\mathcal{I}(M) \subseteq \mathcal{I}(P)$ and extending $M$ in any direction would violate the former constraint. We denote the set of all maximal rectangles of $P$ by $\mathcal{M}(P)$.

Given a polygon $P$, a set of rectangles $\mathcal{R}$ is a rectangle cover (or cover for short) if (i) for each $R \in \mathcal{R}, \mathcal{I}(R) \subseteq \mathcal{I}(P)$ and (ii) $\mathcal{I}(P)=\bigcup_{R \in \mathcal{R}} \mathcal{I}(R)$. A cover is a partition if the pair-wise intersections of the rectangles contain no inner points. For a set of rectangles $\mathcal{R}$ and rectangle $R$ that is not necessarily part of $\mathcal{R}$, we use the shorthand notation $\mathcal{R}_{\supseteq R}:=\left\{R^{\prime} \in \mathcal{R}: \mathcal{I}\left(R^{\prime}\right) \supseteq \mathcal{I}(R)\right\}$.

[^0]

Figure 3: Two polygons (without holes) and their partition into base rectangles. The left shows that the number of base rectangles can be significantly smaller than the number of grid rectangles, whereas both coincide on the right.

We study the Weighted Rectangle Cover problem (WRC) as follows: Given a (non-self-intersecting, rectilinear) polygon $P=(E, \mathcal{H})$, as well as $\alpha, \beta \in \mathbb{R}_{\geq 0}$, find a rectangle cover $\mathcal{C}$ of $P$ that minimizes $\sum_{R \in \mathcal{C}} c_{\alpha, \beta}(R)$, where the cost of a rectangle $R$ is $c_{\alpha, \beta}(R):=\alpha+\beta \cdot \mathcal{A}(R)$.

Note that ratio $\frac{\beta}{\alpha}$ allows for a smooth transition between the objective of the $\mathcal{N} \mathcal{P}$ hard unit-weight Rectangle-Cover problem $(\beta=0)$ and the Rectilinear Polygon Partition problem $(\beta \rightarrow \infty)$, which can be solved in polynomial time, see Figure 1 for examples.

## 4 Base Rectangles and Rectangle Powersets

In this section, we introduce a coarse discretization into base rectangles and prove that there is always an optimal WRC solution that can be derived from them. We show that such discretizations can be computed efficiently, which incidentally allows us to improve upon pixel-based and grid-based approaches for the unit-weight case. Further, their use leads to speedups for various algorithms for the weighted and unweighted problem on real-world instances, which we empirically provide evidence for in Section 6.

Before we introduce our new concept of base rectangles, we give the definition of grid rectangles, also known as Hanan grid.

Definition 4.1 (Grid Rectangles [3]). Extend each edge $e \in \mathcal{E}(P)$ of a polygon $P$ to infinity on both ends and consider all intersections of all these lines. This partitions $\mathcal{I}(P)$ into a set of grid rectangles $\mathcal{D}(P)$.

Definition 4.2 (Base Rectangles). From each concave vertex $v \in \mathcal{V}(P)$ of a polygon $P$, extend one horizontal and one vertical ray from $v$ through $\mathcal{I}(P)$ until it meets with an edge in $\mathcal{E}(P)$. This partitions $\mathcal{I}(P)$ into a set of base rectangles $\mathcal{B}(P)$.

Note that grid rectangles are a refinement of base rectangles. Figure 3 shows two examples with subdivisions of a polygon into base rectangles, which in one case coincides with a subdivision into grid rectangles, but makes a large difference in the other. A polygon with only integral vertices can always be partitioned into a set of pixels and its set of base rectangles (grid rectangles) is a coarsening of this pixelwise partition.

Lemma 4.1. Given a polygon $P$ with $2 n$ edges, the number of base rectangles $|\mathcal{B}(P)|$ is $O\left(n^{2}\right)$ in the worst-case.

Proof. As $|\mathcal{V}(P)| \leq 2 n$, we draw at most $2 n$ horizontal and $2 n$ vertical lines. Each base rectangle is uniquely identified by its top left vertex, which is formed by the intersection of a horizontal and a vertical line. Thus, there are at most $2 n \cdot 2 n=4 n^{2}$ base rectangles.

Lemma 4.2. Given a polygon $P$ with $2 n$ edges, the base rectangles $\mathcal{B}(P)$ can be computed in $O(n \log n+|\mathcal{B}(P)|)$ time.

Proof. We first compute the arrangement, using the well-known sweep line algorithm, in $O(n \log n)$ time. Within the same time bound, we can build a vertical ray-shooting data structure for the arrangement, and one for horizontal ray-shooting. Based on those, one can determine the vertical, and horizontal, extension ray that emits from a concave vertex in $O(\log n)$ time, taking $O(n \log n)$ total time. Finally, we compute the $O(|\mathcal{B}(P)|)$ sized arrangement, which includes the intersection points of the rays' line segments, in $O(n \log n+|\mathcal{B}(P)|)$ time, using [6] for example.

Definition 4.3 (Rectangle Powerset). For a set of rectangles $\mathcal{R}$, the powerset $\Gamma(\mathcal{R})$ is the set of all rectangles that can be covered by a subset from $\mathcal{R}$.

Figure 4 shows an example of a rectangle powerset that is generated by three rectangles. Recall that for the unit-weight rectangle cover problem, it is well known that there is always an optimal solution that consists of maximal rectangles [13], and that the set of maximal rectangles $\mathcal{M}(P)$ has size $O\left(n^{2}\right)$ in the worst-case. Observe that the powerset of the base rectangles $\Gamma(B(P))$ always contains the maximal rectangles $\mathcal{M}(P)$. Though our rectangle powersets can be much larger, their size remains polynomial.

Lemma 4.3. Given a set of $r$ rectangles $\mathcal{R},|\Gamma(\mathcal{R})| \in O\left(r^{2}\right)$.
Proof. Let $V$ be the set of all top left and bottom right vertices of rectangles in $\mathcal{R}$. Then, $|V| \leq 4 r$. Each rectangle that can be covered by a subset of $\mathcal{R}$ must have its top left and bottom right vertex in $V$, so at most $\binom{4 r}{2} \in O\left(r^{2}\right)$ different rectangles can be covered by $\mathcal{R}$.

Corollary 4.1. Given a polygon $P$ with $2 n$ edges, the rectangle powerset of its base rectangles contains at most $|\Gamma(\mathcal{B}(P))| \in O\left(|\Gamma(\mathcal{B}(P))|^{2}\right) \subseteq O\left(n^{4}\right)$ rectangles.

We now show that it suffices to only consider base rectangles as candidates for an optimal cover in the weighted problem setting, which also constitutes the main result of this section. The following result is well-known for the unit-weight case, but its correctness depends on the cost function for the general case. We show that it holds for our cost functions and use it as building block for our main result in Theorem Theorem 4.1. (Similar proofs [32, 15] are known for other problems.)

Lemma 4.4 (Grid-aligned). For every polygon $P$ and reals $\alpha, \beta \geq 0$, there exists an optimal weighted rectangle cover $\mathcal{C}$ such that $\mathcal{C} \subseteq \Gamma(\mathcal{D}(P)$ ), where $\mathcal{D}(P)$ denotes the set of grid cells defined by the vertices of $P$.


Figure 4: A set of three interior-disjoint rectangles (far left) and all five rectangles in its rectangle powerset.

Proof. Let $\Pi_{x}(Q)$ be the set of (distinct) $x$-coordinates of a set of points $Q$, and $\Pi_{y}(Q)$ its set of $y$-coordinates. Recall that the vertices of the Hanan-grid $\mathcal{D}(P)$ are given by the Cartesian product $\Pi_{x}(\mathcal{V}(P)) \times \Pi_{y}(\mathcal{V}(P))$.

Using a sweep argument with a vertical line, we will show that, for any cover $\mathcal{C}^{\prime \prime}$, there exists a cover $\mathcal{C}^{\prime}$ with $\Pi_{x}\left(\mathcal{V}\left(\mathcal{C}^{\prime}\right)\right) \subseteq \Pi_{x}(\mathcal{V}(P))$ and cost no larger than $\mathcal{C}^{\prime \prime}$. Using the symmetric sweep argument with a horizontal line on $\mathcal{C}^{\prime}$ shows the lemma. Note that this argument applies in particular to an optimal cover $\mathcal{C}^{\prime \prime}$.

The sweep argument proceeds with ascending $x$-coordinates and maintains a cover $\mathcal{C}^{\prime}$ that remains feasible at all times, i.e. at all $x$-events. Initially, we set $\mathcal{C}^{\prime}:=\mathcal{C}^{\prime \prime}$ and maintain the sweep-invariant that all vertices of $\mathcal{C}^{\prime}$ that are to the left of the sweep line are contained in $\Pi_{x}(\mathcal{V}(P))$. It remains to specify how to update $\mathcal{C}^{\prime}$ at the $x$-events, i.e. for all $x$-values in $\Pi_{x}\left(\mathcal{V}\left(\mathcal{C}^{\prime \prime}\right)\right) \backslash \Pi_{x}(\mathcal{V}(P))$.

For a given $x$-event, consider all rectangles in the current cover $\mathcal{C}^{\prime}$ that have a vertical edge that is contained in the sweep line. Let $h^{-}$denote the total (summed) height of all incident rectangles that are to the left of the sweep line, and $h^{+}$the total height of all incident rectangles that are to its right. Now, jointly moving all vertical sides of those rectangles by a small $\delta \geq 0$, i.e. to the right, or by a small $\delta<0$, i.e. to the left, changes the cost of the rectangle set $\mathcal{C}^{\prime}$ exactly by

$$
\begin{equation*}
\beta \cdot \delta\left(h^{-}-h^{+}\right) . \tag{1}
\end{equation*}
$$

Thus, there is one direction that does not increase cost. Note that this is true for all $\delta \in\left[x^{-}-x, x^{+}-x\right]$, where $\left|x^{-}-x\right|$ is the distance of the sweep line to its predecessor value in $\Pi_{x}(\mathcal{V}(P))$ and $x^{+}-x$ is the distance of the line to its successor value in $\Pi_{x}\left(\mathcal{V}\left(\mathcal{C}^{\prime \prime}\right)\right) \cup$ $\Pi_{x}(\mathcal{V}(P))$. Clearly, deleting all rectangles $R$ from $C^{\prime}$ that become degenerate by moving, i.e. $\mathcal{A}(R)=0$, does not increase cost, since $\alpha \geq 0$. It remains to show that the set of rectangles $\mathcal{C}^{\prime}$ remains a valid cover of the polygon interior for those values of $\delta$.

To see that the cover remains feasible when moving the boundaries to the right ( $\delta \geq 0$ ), consider an arbitrary incident rectangle $R$ that lies to the right of $x$. ( $R$ is shortened by the move.) If its left boundary left $(R)$ is covered by rectangles that are to the left of the sweep line and incident to it, then there is nothing to show. If there is a point $q \in \operatorname{left}(R)$ that is not covered by a rectangle that is to the left of the sweep line and incident to it, then we have that $q$ is in the interior of the polygon. This is due to the $x$-events being from $\Pi_{x}\left(\mathcal{V}\left(\mathcal{C}^{\prime \prime}\right)\right) \backslash \Pi_{x}(\mathcal{V}(P))$ only, i.e., those $x$-coordinates of $\mathcal{C}^{\prime \prime}$ that are different from those of the polygon. Let $q^{\prime} \in \mathcal{I}(P)$ have the same $y$-coordinate as $q$, but an infinitesimally smaller $x$-coordinate. Since the initial set $\mathcal{C}^{\prime \prime}$ is a cover of the polygon's interior $\mathcal{I}(P)$, there is a rectangle $R^{\prime} \in \mathcal{C}^{\prime}$ that is not incident to the sweep line and contains $q^{\prime} \in \mathcal{I}\left(R^{\prime}\right)$. Thus, $q \in \mathcal{I}\left(R^{\prime}\right)$ and remains covered. The argument for moving the vertical boundaries to the left $(\delta<0)$ is analog.

Using the symmetric sweep argument with a horizontal line on $\mathcal{C}^{\prime}$ shows the lemma.
Theorem 4.1 (Base-Aligned). For every polygon $P$ and reals $\alpha, \beta \geq 0$, there exists an optimal weighted rectangle cover $\mathcal{C}$ such that $\mathcal{C} \subseteq \Gamma(\mathcal{B}(P))$.

Proof. From Lemma 4.4, we have that there is an optimal solution $\mathcal{C}^{\prime \prime}$ whose rectangles consist of vertices from the Hanan-grid $\mathcal{D}(P)$.

Using a sweep argument with a vertical line, we will show that, for any such cover $\mathcal{C}^{\prime \prime}$, there exists a cover $\mathcal{C}^{\prime}$ with, cost no larger than $\mathcal{C}^{\prime \prime}$, whose rectangles have vertical sides that are either contained in the boundary of $P$ or in the ray of a concave vertex. Using the symmetric sweep argument with a horizontal line shows the theorem.

As in the previous proof, our invariant will be that all vertical boundaries that are to the left of the sweep line have this property. The $x$-events of the sweep line are however from the $\operatorname{grid} \Pi_{x}(\mathcal{V}(P))$.

For a given $x$-event, consider the vertical chords through the interior $\mathcal{I}(P)$ with the given $x$-coordinate (i.e. the intersection of the sweepline and $\mathcal{I}(P)$ ). Note that chords are either separated by a section of the boundary or by the exterior of polygon $P$. For a given chord, consider all rectangles of $\mathcal{C}^{\prime}$ that have a vertical boundary contained in the chord. If the chord contains a concave vertex of the boundary of $P$, the incident boundaries of the rectangles are already as desired. For the case that the chord does not contain a concave vertex, we observe that this is only possible if the chord contains no vertex of the polygon $\mathcal{V}(P)$ at its top, or at its bottom. Thus, all rectangles whose vertical sides are incident with the chord can be moved to the predecessor or successor value in $\Pi_{x}(\mathcal{V}(P))$ without increasing the cost. The change in cost of $\mathcal{C}^{\prime}$ for moving, to the left or right, is again given by Equation 1. Thus, moving in one of the directions has non-increasing cost. Moreover, since the chord at the predecessor $x$-coordinate has the identical $y$-range, all vertical boundaries that are incident in the chord may be moved to the left until there is a concave vertex on the chord. We apply this argument separately to each of the vertical chords that are contained in the sweep line. After all $x$-events are resolved, we have obtained a cover $\mathcal{C}^{\prime}$ with the stated property for vertical boundaries.

Lemma 4.5. Let $A_{\min }$ be the minimum area of any base rectangle of a given polygon $P$. Then, a partition is an optimal cover of $P$ with cost function $c_{\alpha, \beta}$ for $\alpha, \beta \geq 0$ if $\beta \cdot A_{\text {min }} \geq \alpha$.

Proof. By Theorem Theorem 4.1, $P$ has an optimal cover $\mathcal{C}$ for cost function $c_{\alpha, \beta}$ such that $\mathcal{C} \subseteq \Gamma(\mathcal{B}(P))$. As $\mathcal{C}$ is optimal, no edge of a rectangle $R \in \mathcal{C}$ is fully contained in another rectangle $R^{\prime}$, otherwise, $R$ could be removed from $\mathcal{C}$ or its area reduced to avoid an overlap with $R^{\prime}$ entirely (see also Section 5.5).

If $\mathcal{C}$ is not a partition, there are at least two rectangles in $\mathcal{C}$ with an overlap of at least $A_{\text {min }}$. Removing that overlap by subdividing one of the two rectangles creates one new rectangle and thus changes the cost of the cover by $\alpha-\beta \cdot A_{\text {min }} \leq 0$. Repeating this step until no overlap remains results in a partition $\mathcal{C}^{\prime}$ of cost no larger than that of $\mathcal{C}$.

To simplify the description of our algorithms, we introduce the base rectangle graph $G^{*}(P)$ of a polygon $P$, where each node $B$ is a base rectangle and has up to four labelled neighbors left $\ell(B)$, top $t(B)$, right $r(B)$, bottom $b(B)$, which are the base rectangles that $B$ shares its left, top, right, or bottom edge with, respectively. We say that a path in $G^{*}(P)$ is unidirectional if it starts at some node and continues only via either left, or top, or right, or bottom neighbors.

Lemma 4.6. Given a polygon $P$ with $2 n$ edges, the base rectangle graph $G^{*}(P)$ can be constructed in $O(n \log n+|\mathcal{B}(P)|)$ time and has $O(|\mathcal{B}(P)|)$ nodes. There are at most $O(n)$ nodes that have less than four neighbors and each unidirectional path has length $O(n)$.

Proof. The construction and the number of nodes is directly implied by Lemma 4.2. Each node that does not have four neighbors must be a base rectangle where at least one of its sides is a segment of an edge in $\mathcal{E}(P)$. By construction (cf. proof of Lemma 4.1), each vertex can cause at most one subdivision of a horizontal and one subdivision of a vertical edge. As there are $\leq 2 n$ vertices, the total number of horizontal and vertical
edge segments is $\leq 2 n$ each. The same argument implies that each unidirectional path has length $\leq 2 n$.

Lemma 4.7. Given the base rectangle graph $G^{*}(P)$, the powerset $\Gamma(\mathcal{B}(P))$ can be reported in $O(|\Gamma(\mathcal{B}(P))|)$ time.

Proof. We first compute the base rectangle graph $G^{*}(P)$ using Lemma 4.6.
For each node $B$ of $G^{*}$, we enumerate all rectangles $R$ that have $\Gamma(R)=\Gamma(B)$ by starting with $b(B)$ and first moving to the respective bottom neighbor in each step for as long as possible. In each step, report the rectangle with top left vertex $\Gamma(B)$ and bottom right vertex equal to the bottom right vertex of the current base rectangle. Let $\min _{y}$ be the $y$-coordinate of the bottom right vertex of the last base rectangle. Repeat the same process starting with $r(B)$, until either a base rectangle whose bottom right vertex has $y$-coordinate $\min _{y}$ is reached or the current base rectangle has no bottom neighbor. In the second case, update $\min _{y}$ to the $y$-coordinate of this last base rectangle. Repeat the process with $r(r(B)), r(r(r(B)))$ and so on until a node without a right neighbor is reached.

## 5 Algorithms

In this section, we present and analyze three heuristic base algorithms and six postprocessing routines to solve the WRC problem quickly in practice. In addition, we give an exact approach in form of a binary linear program, which we also use to evaluate the solution qualities in our experimental evaluation.

### 5.1 Binary LP (BIP)

Our binary linear program is based on the formulation used for rectilinear picture compression [18], which essentially solves the unweighted rectangle cover problem (cf. Section 2). Recall that a crucial difference between the unweighted and the weighted problem is that we cannot restrict ourselves to consider only maximal rectangles as candidates for the cover. Our formulation is based on base rectangles and reads as follows:

$$
\begin{array}{lll}
\min & \sum_{R \in \Gamma(\mathcal{B}(P))} x_{R} \cdot c_{\alpha, \beta}(R) & \\
\text { s.t. } & \sum_{R \in \Gamma(\mathcal{B}(P))_{\supseteq B}} x_{R} \geq 1 & \forall B \in \mathcal{B}(P) \\
& x_{R} \in\{0,1\} & \forall R \in \Gamma(\mathcal{B}(P)) \tag{4}
\end{array}
$$

We have one binary variable $x_{R}$ per element of $\Gamma(\mathcal{B}(P))$ and one constraint per base rectangle, where the constraints in (3) ensure that every base rectangle is covered by at least one selected rectangle. By Corollary 4.1 and Lemma 4.1, we thus have $O\left(n^{4}\right)$ variables and $O\left(n^{2}\right)$ constraints.

Let $\mathcal{M}(P)$ be the set of maximal rectangles of $P$. To obtain a formulation for the unweighted rectangle cover problem, it suffices to only have a binary variable for each rectangle in $\mathcal{M}(P)$. In addition, we set a unit cost function in (2) and replace the constraints (3) by

$$
\sum_{M \in \mathcal{M}(P)_{\supseteq B}} x_{R} \geq 1 \quad \forall B \in \mathcal{B}(P)
$$

Besides giving a formulation for polygons where not all vertices are integral, the number of constraints also remains $O\left(n^{2}\right)$ and thus only depends polynomially on the size of the polygon (i.e., the number of edges), as opposed to its area in the original formulation [18].

Note that using $\mathcal{M}(P)$ instead of $\Gamma(\mathcal{B}(P))$ for the weighted problem leads to incorrect solutions (e.g. Figure 1).

### 5.2 Greedy Weighted Set Cover

Our first heuristic is an adaptation of the greedy weighted set cover algorithm first described by Chvatal [7]. Similar to the binary LP, it uses the rectangle powerset of all base rectangles of a given polygon $P$ as candidate set and seeks to ensure that all base rectangles are covered. The algorithm incrementally computes a cover $\mathcal{C}$ as follows: Let $\mathcal{I}(\mathcal{C})=\bigcup_{R \in \mathcal{C}} \mathcal{I}(R)$. For each rectangle $R \in \Gamma(\mathcal{B}(P))$, the algorithm maintains its effective area $a(R)$ as the area of $\mathcal{I}(R) \backslash \mathcal{I}(\mathcal{C})$. Furthermore, the algorithm keeps track of all base rectangles that are currently uncovered. Note that as the rectangle powerset provides the candidates to build $\mathcal{C}$ and two base rectangles can share at most a common edge but never properly intersect, a base rectangle is either fully covered or fully uncovered, but never partially covered. Among all candidate rectangles, the algorithm chooses a rectangle $R$ that minimizes the relative effective cost $\frac{c(R)}{a(R)}$, adds it to $\mathcal{C}$, and removes it from the set of candidates. Afterwards, it updates the set of uncovered base rectangles, as well as the effective area of all candidates, removing any candidate with an effective area of 0 . The procedure is repeated until no candidates are left.

Lemma 5.1. The Greedy Weighted Set Cover algorithm runs in $O\left(|\mathcal{B}(P)|^{3}\right) \subseteq O\left(n^{6}\right)$ time, uses $O\left(|\mathcal{B}(P)|^{2}\right) \subseteq O\left(n^{4}\right)$ space on a polygon $P$ with $2 n$ edges, and returns an $O(\log n)$-approximate solution for the WRC problem.

Proof. Let $b:=|\mathcal{B}(P)|$ be the number of base rectangles of $P$. Computing the initial candidate set and initializing the respective effective areas takes $O\left(b^{2}\right)$ time by Corollary 4.1. In each iteration of the algorithm, choosing the candidate rectangle $R$ takes $O\left(b^{2}\right)$ time if we keep all candidates in a list ${ }^{2}$ and search it linearly. We keep the set of uncovered base rectangles in a hash table and assume that updates and queries can be handled in (expected) constant time. Obtaining the set of newly covered base rectangles takes time $O(b)$. Let $r_{i}$ be the number of newly covered base rectangles in iteration $i$. To update the effective area of a candidate rectangle, we test whether it covers any of the newly covered base rectangles in $O\left(r_{i}\right)$ time and, if necessary, update its relative effective cost in constant time. As there are $O\left(b^{2}\right)$ candidates, this takes $O\left(b^{2} \cdot r_{i}\right)$ in the $i$-th iteration. Each iteration covers at least one base rectangle, thus there are at most $O(b)$ iterations, yielding a total running time of $O\left(b \cdot b^{2}+b \cdot b+b^{2} \cdot \sum_{i} r_{i}\right)=O\left(b^{3}\right)$, as each base rectangle can only be newly covered once.

We need to maintain $O\left(b^{2}\right)$ candidates and monitor $O(b)$ base rectangles, so the space complexity is $O\left(b^{2}\right)$. The $O\left(n^{6}\right)$ running time and $O\left(n^{4}\right)$ space follow by Lemma 4.1.

### 5.3 Strip Cover

Our next algorithm is based on the strip cover algorithm by Kumar and Ramesh [3] (cf. Section 2). In the original description, the algorithm uses grid rectangles and does not state any running times. Whereas the adaptation to the base rectangle graph is straightforward, we also describe how to implement the algorithm efficiently: We first

[^1]obtain the base rectangle graph $G^{*}(P)$ of the input polygon $P$. For each node $B$ of $G^{*}(P)$, let $h(B)$ denote the height of $B$, which is defined as the length of the longest unidirectional "bottom-going" path that starts with the edge to its bottom neighbor, $\{B, b(B)\}$. Thus, if $B$ has no bottom neighbor, $h(B)=0$. Now, for each node $B$ that does not have a top neighbor $t(B)$, follow the unidirectional "left-going" path starting with the edge $\{B, \ell(B)\}$ until a node $B^{\prime}$ is reached that either has no left neighbor or $h\left(\ell\left(B^{\prime}\right)\right)<h(B)$. Symmetrically, follow the unidirectional "right-going" path starting with the edge $\{B, r(B)\}$ until a node $B^{\prime \prime}$ is reached that either has no right neighbor or $h\left(r\left(B^{\prime \prime}\right)\right)<h(B)$. From $B^{\prime \prime}$, follow the unidirectional "bottom-going" path starting with $\left\{B^{\prime \prime}, b\left(B^{\prime \prime}\right)\right\}$ and stop after $h(B)$ steps. Let $B^{\prime \prime \prime}$ be the last node of this path. Report the rectangle $R$ with $\Gamma(R)=\Gamma\left(B^{\prime}\right)$ and $\left.\lrcorner(R)=\right\lrcorner\left(B^{\prime \prime \prime}\right)$. The set of all rectangles $R$ obtained in this way, eliminating duplicates, is returned as cover. Note that by construction, each rectangle $R$ is maximal.

Lemma 5.2. The greedy strip cover algorithm runs in $O\left(n^{2}\right)$ time, uses $O(|\mathcal{B}(P)|)$ space on a polygon $P$ with $2 n$ edges, and returns a cover of size $O(n)$.

Proof. We first compute the base rectangle graph $G^{*}(P)$ in $O(n \log n+|\mathcal{B}(P)|)$ time by Lemma 4.6. Computing $h(B)$ for each node $B$ requires one traversal of the graph and can thus be done in time linear in the size of $G^{*}(P)$. Again by Lemma 4.6, the number of base rectangles without top neighbor is in $O(n)$ and each unidirectional path has length $O(n)$. Hence, for each starting node $B$, finding $B^{\prime}, B^{\prime \prime}, B^{\prime \prime \prime}$ takes $O(n)$ time.

The algorithm adds at most one rectangle to the cover for each base rectangle without top neighbor, thus the size of the returned cover is in $O(n)$. Testing whether the resulting rectangle has already been found earlier hence takes no more $O(n)$ time, which yields a total running time of $O\left(n^{2}\right)$.

Storing $G^{*}(P)$ requires $O(|\mathcal{B}(P)|)$ space. As we compute a cover of cardinality $O(n)$, we need at most $O(n)$ space to store the result and implement the duplicate elimination.

As mentioned above, the cover computed by the algorithm is a subset of all maximal rectangles $\mathcal{M}(P)$, but $|\mathcal{M}(P)|$ can be as large as $O\left(n^{2}\right)$ [13].

Note that the Strip Cover algorithm does not consider any cost function. To make it "cost-aware", we will combine it with postprocessors, which are described in Section 5.5.

### 5.4 Partition Algorithm

For reasons of self-containedness, we here briefly review Ohtsuki's optimal rectangle partitioning algorithm [28] before we describe our adaptations for the rectangle cover problem, which are implemented as postprocessing routines and described in Section 5.5.4.

The algorithm computes a partition by first locating pairs of concave vertices that can be connected by a straight horizontal or vertical segment in a given polygon $P$ 's interior. Such a segment is also called degenerate diagonal or chord. Construct a graph with a node for each chord and an edge between two nodes if the corresponding chords properly cross each other in the interior. As two horizontal (two vertical) chords never cross, the graph is bipartite. Next, compute a maximum-cardinality independent set $I$, which can be done in polynomial time on bipartite graphs via matching. The chords in $I$ partition $P$ into smaller polygons. For each concave vertex in the remaining, smaller polygons, extend a horizontal (or vertical) line into the interior until it hits an edge of the polygon. The result is an optimal partition of $P$ into a set of rectangles.

Lemma 5.3 ([19]). The partition algorithm can be implemented to run in time $O\left(n^{1.5} \log n\right)$ on a polygon with $2 n$ edges.

### 5.5 Postprocessing

Recall that neither the Strip Cover algorithm (Section 5.3) nor the Partition algorithm (Section 5.4) use rectangle weights to compute a cover. We therefore introduce four postprocessing routines that connect the aforementioned algorithms to the WRC problem and also serve to improve the initial solution by the Greedy Set Cover algorithm (Section 5.2). All postprocessors assume a given initial cover $\mathcal{C}$ and can be chained.

### 5.5.1 Prune

This postprocessor tests for each rectangle $R \in \mathcal{C}$ whether all base rectangles contained in it are also covered by at least one other rectangle in the cover. If this is the case, rectangle $R$ is removed from $\mathcal{C}$ and the algorithm continues with the next rectangle.

Lemma 5.4. The prune postprocessor runs in time $O(|\mathcal{C}| \cdot|\mathcal{B}(P)|)$ for a polygon $P$ and an initial cover $\mathcal{C}$.

Proof. We first compute the base rectangle graph $G^{*}(P)$ in $O(n \log n+|\mathcal{B}(P)|)$ time by Lemma 4.6. We maintain a counter for each base rectangle $B \in \mathcal{B}(P)$ that stores the number of cover rectangles it is contained in. To initialize the counters, for each rectangle $R \in \mathcal{C}$, we iterate over all base rectangles that are fully contained in $R$ and increase their counter. We then iterate a second time over all $R \in \mathcal{C}$, check each for redundancy using the counters, and update the counters where necessary. The total running time hence is $O(|\mathcal{C}| \cdot|\mathcal{B}(P)|)$.

### 5.5.2 Trim

Similar to the prune postprocessor, trim checks for redundancies, however w.r.t. the area of a rectangle in the cover. For each rectangle $R \in \mathcal{C}$, trim identifies the set of base rectangles $U_{R}$ that are only contained in and thus covered by $R$. It then replaces $R$ in $\mathcal{C}$ by the bounding box of these base rectangles, $\mathbb{B}\left(U_{R}\right)$. Trim can be implemented analogously to prune.

Lemma 5.5. The trim postprocessor runs in time $O(|\mathcal{C}| \cdot|\mathcal{B}(P)|)$ for a polygon $P$ and an initial cover $\mathcal{C}$.

### 5.5.3 Rectangle Splits

The idea behind rectangle splits is to eliminate rectangles that have large overlap with other rectangles in the cover, but cannot be pruned or trimmed any further. To split a rectangle $R$, we first remove it from the cover. As $R$ could not be trimmed any further, this results in a non-empty set of uncovered base rectangles $U_{R}$. For each polygon formed by a maximally connected subset of $U_{R}$, called gap, the rectangle split postprocessors try to newly cover the gap using different approaches. The split is accepted if the total cost of the new cover of $U_{R}$ is less than the cost of $R$, and rejected and undone otherwise.

Lemma 5.6. For a set of gaps formed by the uniquely covered base rectangles $U_{R}$ of a rectangle $R$ within a cover $\mathcal{C}$ for a polygon $P$, the total number of vertices of all gaps is in $O\left(|\mathcal{C}|^{2}\right)$.

Proof. The polygon $Q$ resulting from the removal of $R$ from $\mathcal{C}$ is the union of $|\mathcal{C}|-1$ rectangles. As the union complexity of a set of rectangles is quadratic in the size of the set [21], both $P$ and $Q$ have $O\left(|\mathcal{C}|^{2}\right)$ vertices.

Each vertex of a gap is either a vertex of $Q, R$, or lies on the intersection of an edge of $R$ and $Q$ each, making it a vertex of $P$. Hence, the total number of vertices of all gaps is $O\left(|\mathcal{C}|^{2}\right)$.

Bounding Box Split. This postprocessor simply covers each gap by its bounding box. As each gap originally was contained in a larger rectangle that was part of the cover, the bounding box of a gap must be fully contained in the polygon's interior.

Lemma 5.7. The bounding box split postprocessor runs in time $O\left(|\mathcal{C}|^{3}\right)$ for a polygon $P$ and a cover $\mathcal{C}$.

Proof. By Lemma 5.6, the total number of vertices for all gaps resulting from the removal of a single rectangle from the cover is in $O\left(|\mathcal{C}|^{2}\right)$. Hence, for each rectangle, there are at most $O\left(|\mathcal{C}|^{2}\right)$ gaps to cover, resulting in a total running time of $O\left(|\mathcal{C}|^{3}\right)$.

Partition Split. As the name suggests, this postprocessor covers each gap using the partition algorithm (cf. Section 5.4).
Lemma 5.8. The partition split postprocessor runs in time $O\left(|\mathcal{C}|^{4} \log |\mathcal{C}|\right)$ for a polygon $P$ and a cover $\mathcal{C}$.

Proof. By Lemma 5.6, the total number of vertices for all gaps resulting from the removal of a single rectangle from the cover is in $O\left(|\mathcal{C}|^{2}\right)$. Hence, computing a partition for all gaps can be done in $O\left(|\mathcal{C}|^{3} \log |\mathcal{C}|\right)$ time per removed rectangle by Lemma 5.3, and thus in $O\left(|\mathcal{C}|^{4} \log |\mathcal{C}|\right)$ time overall.

### 5.5.4 Rectangle Joins

We now describe two postprocessors that try to replace a set of rectangles contained in the cover by a single rectangle if this improves the cost of the cover. The postprocessors differ in the set of rectangles they consider for a join. Joining a set of rectangles $\mathcal{R} \subseteq \mathcal{C}$ removes $\mathcal{R}$ from $\mathcal{C}$ and inserts the bounding box of the set $\mathbb{B}(\mathcal{R})$.

Simple Join. For the simple or aligned join, the algorithm only considers sets of rectangles where all rectangles have the same minimum and maximum $y$-coordinate, i.e., they are horizontally aligned, or sets of rectangles where all rectangles have the same minimum and maximum $x$-coordinate (vertically aligned).

Lemma 5.9. The simple aligned join postprocessor runs in time $O(|\mathcal{C}| \cdot n)$ for a polygon $P$ with $2 n$ edges and a cover $\mathcal{C}$.

Proof. We first iterate over all $R \in \mathcal{C}$ and sort each into two buckets identified by a pair of $x$ - or $y$-coordinates, respectively, according to the minimum and maximum $x$ - or $y$-coordinates of $R$. We assume that the rectangles in a bucket do not overlap, i.e., if necessary $\mathcal{C}$ has been pruned (cf. Section 5.5.1) and trimmed (cf. Section 5.5.2) before. Thus, each bucket contains at most $O(n)$ rectangles by Lemma 4.6 (longest unidirectional path). We sort each bucket identified by $y$-coordinates according to the $x$-coordinates of its rectangles, and vice-versa. The test for joins can then be implemented as a sweep
line process, where a rectangle $R$ is joined with its predecessor rectangle $R^{\prime}$ if and only if the join leads to a reduction in cost and the joined rectangle is fully contained in $\mathcal{I}(P)$ by checking whether a straight diagonal line from $\Gamma\left(R^{\prime}\right)$ to $\lrcorner(R)$ intersects with an edge of the polygon in $O(n)$ time. The sweep line processes take $O(|\mathcal{C}| \cdot n)$ total time and dominate the rest.

Full Join. The postprocessor works similar to the simple aligned join, but considers joins of arbitrary sets of rectangles.

Lemma 5.10. The full join postprocessor runs in time $O\left(|\mathcal{C}|^{2} \cdot n\right)$ for a polygon $P$ with $2 n$ edges and a cover $\mathcal{C}$.

Proof. Fix an arbitrary order of the rectangles in $\mathcal{C}$. For each $R \in \mathcal{C}$, test whether a join with any of its successors improves the cost of the cover and yields a rectangle that is fully contained in $\mathcal{I}(P)$. If this is the case, the joined rectangle takes the place of $R$ in the order and the algorithm tries to join it with further successors. Checking whether the joined rectangle is contained in $\mathcal{I}(P)$ can be accomplished in $O(n)$ time by testing for intersections with the polygon's edges. This can be necessary at most $O\left(|\mathcal{C}|^{2}\right)$ times.

## 6 Experiments

We implemented ${ }^{3}$ all algorithms described in Section 5 in C++17 and compare them against each other on a large and diverse set of instances, having 186837 polygons in total. To the best of our knowledge, this is the first experimental evaluation of algorithms for the WRC problem. For the smaller instances, we include exact solutions based on the ILP from Section 5.1. It would have been interesting to compare our algorithms also to the greedy set cover-based algorithm by Heinrich-Litan and Lübbecke [18] for the unweighted case. Unfortunately, the code is no longer available.

### 6.1 Methodology

We compiled our code with GCC $11.4^{4}$ and used Gurobi $10.02^{5}$ as ILP solver. To load instances and for geometric computations, we rely on the Computational Geometry Algorithms Library [30]. All our implementations are deterministic and single-threaded, but Gurobi is inherently multi-threaded. For this reason, we do not compare the running times of the ILP-based algorithm with the others, but only the solution quality. All runtime experiments were conducted on a machine with an $\operatorname{Intel}(\mathrm{R}) \mathrm{Xeon}(\mathrm{R}) \mathrm{E} 5 \mathrm{CPU}$ and 1.5 TB of main memory, running Ubuntu Linux 22.04 (Kernel 5.15). To counteract errors of measurement, every experiment was assigned to an exclusive core and carried out three times. For each algorithm, polygon, and cost function, we report the absolute solution quality, i.e., the total cost of the cover, and median of the absolute running time. Furthermore, for each such triple, we computed the relative solution quality and and relative running time as the ratio of the corresponding absolute value and the best value obtained by any algorithm for this polygon and cost function. We set a timeout of 1 h per polygon for the ILP solver, and 4 h per instance for the other algorithms.

Table 1: Algorithms evaluated experimentally.

| Base Algorithm | Postprocessors | Short Name |
| :---: | :---: | :---: |
| partition | simple join full join | par par-j par-f |
| strip | prune, trim prune, trim, bb-split prune, trim, par-split | ```strip strip-pt strip-ptb strip-pts``` |
| greedy | prune, trim | grdy grdy-pt |
| ilp | - | ilp |

Table 2: Instances used in the evaluation, where $\# P$ : \#non-trivial polygons, $\max \mathcal{H} / \max \mathcal{V} / \max \mathcal{D} / \max \mathcal{B}: \max$. \#holes/vertices/grid rectangles/base rectangles.

| data set | $\# P$ | $\max \mathcal{H}$ | $\max \mathcal{V}$ | $\max \mathcal{D}$ | $\max \mathcal{B}$ | $\max \frac{\mathcal{D}}{\mathcal{B}}$ |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: |
| ccitt | 11970 | 192 | 27478 | 539347 | 159132 | 3.39 |
| icons | 211 | 10 | 108 | 134 | 134 | 1.86 |
| nasa | 13230 | 9804 | 124204 | 1826606 | 1208309 | 3.43 |
| caltech | 22989 | 1929 | 10986 | 55379 | 42982 | 3.69 |
| textures | 2674 | 16249 | 160132 | 825464 | 713421 | 2.94 |
| aerials | 131818 | 24079 | 242330 | 583977 | 879458 | 1.76 |
| dats | 3919 | 14530 | 69740 | 250239 | 235841 | 3.56 |
| cgshop | 26 | 67 | 94778 | 26293767 | 988050 | 145.84 |

### 6.2 Algorithms, Instances, and Cost Functions

Algorithms. We included all algorithms described in Section 5 and combined them with different postprocessing routines. Table 1 gives an overview of the different combinations we evaluated and the shorthand names we use in the discussion of the results.

Instances. We used a diverse collection of instances from experimental evaluations of related problems $[18,22]$. Where necessary, we converted the instances to the $\mathrm{WKT}^{6}$ format first. With the exception of the dats and cgshop instance sets, all sets of instances were retrieved from Koch and Marenco's [22] repository of images they binarized ${ }^{7}$. Even though our algorithms can also handle real-valued polygons, all instances here only use integer coordinates. Our instance sets are:
ccitt: fax test images, originally provided to compare compression algorithms; icons: small black and white icons ranging from $6 \times 6$ to $24 \times 24$ pixels; nasa: binary versions of large satellite images provided by NASA; caltech: binary versions of various standard, medium-sized images used in computer vision research; textures: binary versions of texture images used in image processing research; aerials: binary versions of aerial photographs used in image processing research; dats: black and white images introduced

[^2]in [18]; cgshop: rectilinear instances from the CG:SHOP 2023 competition ${ }^{8}$.
Each instance in these collections is a "multipolygon", i.e., it can consist of multiple polygons. Our algorithms solve each polygon individually and independently. In many cases, hole-free rectangles are among these polygons, which can be covered optimally by a single rectangle. This case is easy to recognize and handle by our preprocessor, as it does not provide any insights into the performance of the main algorithm. We therefore do not consider these trivial polygons in our evaluation any further. To facilitate an evaluation w.r.t. the metrics of a polygon, we report solution quality and running time for each non-trivial polygon in an instance rather than the sum over all polygons in the multipolygon. See Table 2 for a summary of statistics, including the maximum number of grid and base rectangles for each instance set as well as the maximum ratio between them. The average ratio of grid and base rectangles was between 1.01 (textures) and 11.24 (cgshop). The maximum size of the powerset of base rectangles, which is used by grdy and ilp, was between 1793 (icons) and 7901671022 (nasa). In total, our study comprises 186837 non-trivial polygons.

Cost Functions. As the different tradeoffs between rectangle creation cost $(\alpha)$ and rectangle area cost $(\beta)$ mainly depend on the ratio of these two values, we only vary $\alpha$ in our experiments and keep $\beta=1$ fixed. For all instances, we ran experiments with $\alpha \in\{1,10,50,100,500,1000\}$. Recall from Lemma 4.5 that if $\alpha \leq \beta \cdot A_{\min }$, where $A_{\text {min }}$ is the smallest area of any base rectangle, the partition algorithm is optimal. As our instances are integer-valued, the partition algorithm is guaranteed optimal for $\alpha \leq 1$.

### 6.3 Results

We only ran algorithm ilp on polygons from the instance sets with smaller polygons icons, dats, and cgshop. Still, ilp ran into timeouts or was terminated due to insufficient memory on polygons from two dats instances and five cgshop instances. Similarly, algorithm grdy ran into several timeouts and memory allocation failures on polygons from aerials, nasa, textures, and dats, and cgshop. We therefore do not compare grdy on polygons from these instances. The two instances with the largest number of base rectangles on which grdy terminated are from the aerials instance set and have 988050 base rectangles (size of powerset: 1.53 billion). All instance on which grdy terminated have at most 108970 vertices. par-f ran into a timeout for one instance of aerials and one of textures, where $\alpha \geq 500$. As these were the only timeouts, we let it finish nonetheless in order to have a complete set of results for the evaluation.

Solution Quality. Across all instances, par-f gave on average the best solution quality for $\alpha \leq 100$, whereas for $\alpha \geq 100$, strip-ptb and strip-pts exceeded all others, see also Figure 5 (left). On polygons from caltech, cgshop, and dats, par-f yielded on average the best cover for all values of $\alpha$, and was only slightly inferior to strip-pts on aerials for $\alpha>500$. On ccitt, icons, nasa, and textures, strip-pts outperformed par-f already for $\alpha \geq 100$ or less (cf. Figure 6).

The performance of strip-pt, strip-ptb, and strip-pts is similar for large values of $\alpha$, whereas strip-pt and strip-ptb are clearly inferior to strip-pts for smaller values. The solution quality was significantly impaired if strip was used without trim and yielded on average up to $44 \%$ and $23 \%$ heavier covers for small $\alpha$. As $\alpha$ grows, the gap in average

[^3]

Figure 5: Mean relative cost for different values of $\alpha$ across all instances (left), and a performance plot showing on how many polygon/ $\alpha$ pairs each algorithm returned a solution with at least a given relative cover cost (right). In both cases, note the logarithmic x axis.



Figure 6: Mean relative cost for different values of $\alpha$, across all instances where grdy terminated, i.e., icons, ccitt, and caltech (left) and only on icons (right). In both cases, note the logarithmic x axis.
relative solution cost widened for par, and par-j, and par-f, showing that the postprocessors gain in effectiveness for large $\alpha$. Recall that strip and par without postprocessors do not consider the given cost function, hence their inferior performance is not surprising. On instance sets where grdy yielded a result for all polygons, i.e., icons, caltech, and ccitt, both grdy and grdy-pt were not significantly better on average than par-f and strippts and partially also par-j for any value of $\alpha$, and only better than strip-ptb for $\alpha=1$ (cf. Figure 6).

Looking at the maximum relative cost rather than the average, par-j, par-f and strip-pts never exceeded the best solution by more than $76 \%, 66 \%$, and $56 \%$, respectively (cf. Figure 5), and found the best cover in around $90 \%$ of all cases. By contrast, strip and strip-pt produced solutions that are by a factor of 80 and 4.7 , respectively, larger than the best. For $\alpha \geq 50$, strip-pts always had the smallest maximum relative cost when looking at all instance sets. The picture is also similar on each instance set individually, except for icons, where both par-j and par-f outperformed strip-pts for all values of $\alpha$ and never returned a cover with weight $33 \%$ larger than the optimal.

In summary, if worst-case performance is prioritized, strip-pts is the candidate of


Figure 7: Running time vs. number of vertices in the polygon for $\alpha=10$ (left) and $\alpha=100$ (right) .
choice for $\alpha \geq 50$, whereas where average performance is considered, par-f is superior for small values of $\alpha$, followed by par-j. Furthermore, all postprocessing routines proved themselves very effective in increasing the solution quality.

Running Time. The fastest algorithm across all cost functions and instance sets was par with mean relative running times of around 1.00 , very closely followed by par-j (around 1.06). par-f was only slightly slower than par-j except for cgshop, where its mean relative running time was around 2.65. strip, strip-p, and strip-pt on average had a relative running time of 7.7 to 8.0 , whereas strip-ptb and strip-pts were on average by a factor of 10 and 11 slower than par. On those instances where grdy and grdy-pt completed, they had relative running times of around 7.3 (caltech), 3.4 (ccitt), and 1.7 (icons). Similar to strip-pt, the postprocessing of grdy-pt only led to a small increase in running time in comparison to grdy. On ccitt and icons, grdy and grdy-pt were faster on average than strip-ptb and strip-pts.

Only par-f showed a strong dependency between running time and $\alpha$, which generally increased together with $\alpha$ on all instance sets. The strongest increase was observed on cgshop, where the maximum running time for a polygon was 239 s for $\alpha=1$ and 1618 s for $\alpha=1000$. The mean running time rose from 13 s to 83 s . The observed increase is due to the implementation of the full join postprocessor: When considering a join of two rectangles, it first checks whether this would improve the solution, and only then checks whether the joined rectangle actually lies within the polygon's interior. The larger the creation cost $\alpha$, the more often a join reduces the weight of the cover and the second check becomes necessary.

All algorithms finished within 3 ms on icons. Across all polygons, the maximum running time of the par- and strip-based algorithms, except par-f, was between 51 min and 58 min . To asses the empirical running time of the par- and strip-based algorithms further, we grouped the polygons by their number of vertices in steps of 100 and computed the arithmetic mean running time for every algorithm and number of vertices group. The results, depicted in Figure 7 for $\alpha=10$ and $\alpha=100$, show a super-linear dependency of the running time on the number of vertices also in practice and again demonstrate the runtime dependency of par-f on $\alpha$ in practice.

In summary, par-f is the algorithm of choice for small values of $\alpha$. It is very fast and offered the best solution quality in this setting. For larger values of $\alpha$, strip-pts is well-suited if solution quality is important and its higher running time is acceptable,
which should in particular be the case for polygons with fewer vertices. If both $\alpha$ and the polygon are large, our experiments suggest par-j as a good compromise between running time and solution quality.

## 7 Conclusion and Future Work

In this paper, we initiated the study of the Weighted Rectilinear Cover problem. We introduced the concept of base rectangles and demonstrated that our new heuristics are both fast and deliver close-to-optimal results. In particular, they are faster and better than the greedy set cover approximation algorithm. We focused on rectangle cost functions that take a rather specific, yet practical form, and were mainly interested in the solution quality that can be achieved in practice.

A logical next step would be to look into more general cost functions. As long as an optimal solution can be built from base rectangles, our algorithms can be directly applied. Another direction is to focus on the time and memory requirements of algorithms that achieve similar solution costs. A more theoretical question is, if improved, i.e. sublogarithmic, approximation bounds for the WRC problem are possible.

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[^0]:    ${ }^{1}$ Note that a self-intersecting polygon can be represented by a set of non-self-intersecting polygons and that two intersecting holes can be represented by a single, larger hole.

[^1]:    ${ }^{2}$ We could use a heap, but as we need to update all candidates later, this does not pay off.

[^2]:    ${ }^{3}$ https://github.com/WeRecCover/WeRecCover
    ${ }^{4}$ with -03 -march=native -mtune=native
    ${ }^{5}$ https://gurobi.com
    ${ }^{6}$ Well-known text representation of geometry
    ${ }^{7}$ https://drive.google.com/drive/folders/1EPj1w_P8Bgg_86dCzOWJVu3JnFsEbrPO [22]

[^3]:    ${ }^{8}$ https://cgshop.ibr.cs.tu-bs.de/competition/cg-shop-2023/

