

# Practical Box Splines for Reconstruction on the Body Centered Cubic Lattice

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**Abstract**—We introduce a family of box splines for efficient, accurate and smooth reconstruction of volumetric data sampled on the Body Centered Cubic (BCC) lattice, which is the favorable volumetric sampling pattern due to its optimal spectral sphere packing property. First, we construct a box spline based on the four principal directions of the BCC lattice that allows for a linear  $C^0$  reconstruction. Then, the design is extended for higher degrees of continuity. We derive the explicit piecewise polynomial representations of the  $C^0$  and  $C^2$  box splines that are useful for practical reconstruction applications. We further demonstrate that approximation in the shift-invariant space—generated by BCC-lattice shifts of these box splines—is *twice* as efficient as using the tensor-product B-spline solutions on the Cartesian lattice (with comparable smoothness and approximation order, and with the same sampling density). Practical evidence is provided demonstrating that not only the BCC lattice is generally a more accurate sampling pattern, but also allows for extremely efficient reconstructions that outperform tensor-product Cartesian reconstructions.

**Index Terms**—BCC, Box Splines, Discrete/Continuous Representations, Optimal Regular Sampling

## I. INTRODUCTION

In this paper we focus on sampling and representation of volumetric data. Two important issues are dealt with: (1) the choice of the sampling pattern; (2) the signal model that links the discrete to the continuous domain and thus serves to effectively interpolate or approximate the underlying continuous phenomenon. We start by introducing both aspects.

### *Optimal Regular Sampling*

The study of optimal regular sampling patterns is not new [11], [18], [24], [29]. For the class of signals that have an isotropic band-limited (or essentially low-pass) spectrum, the problem of optimal regular sampling can be answered using the solution to the optimal circle (2-D) or sphere (3-D) packing problem. This is due to the fact that the sparsest regular (lattice) distribution of samples in the spatial domain demands the tightest arrangement of the replicas of the spectrum in the frequency domain. Therefore, the optimal sampling lattice is simply the dual of the densest packing lattice. This sampling lattice constitutes the *best generic* lattice for sampling trivariate functions.

Gauß proved that the densest packing in 2-D is obtained by the hexagonal lattice [6]. For the 3-D sphere packing problem, which is also known as the Kepler problem dating from the early 17th century, Gauß proved that the face centered cubic (FCC) lattice attains the highest possible density [6]. Further, the Kepler conjecture—that the FCC packing is an optimal packing

of spheres even when the lattice condition is not imposed—was not proven until 1998 by a lengthy computer-aided proof [15].

For the class of isotropic band-limited signals, the hexagonal lattice can contain 14% more information than the Cartesian lattice without introduction of any aliasing [24]. The analogous 3-D FCC replication of the spectrum allows about 30% more information to be captured without any aliasing in the spectrum of the function. Having the spectrum replicated on the FCC lattice in the frequency domain corresponds to sampling on the body-centered cubic (BCC) lattice in the spatial domain.

Despite their theoretical advantages, optimal sampling strategies have only had a limited impact on practical applications. On the one hand, there are no 3-D scanning devices yet that produce data directly sampled on the BCC lattice; on the other hand, if the scanning machines adopt the optimal sampling pattern, main signal processing tools, such as reconstruction, to process and analyze the data are needed.

### *Reconstruction Kernels*

Reconstruction from sampled data refers to the procedure of interpolating or approximating the underlying continuous-domain signal. Traditionally, the design of reconstruction *filters* is a rich area in signal processing. This approach is discrete-to-discrete and thus provides estimates of the signals on another regular sampling lattice. Typically, constraints in the frequency domain are used to guide the filter design process (e.g., [3], [12], [28]). On the other hand, approximation theory has shown how to design reconstruction *kernels*, which are defined in the continuous domain and allow to estimate the signal at any point in space (e.g., [16], [25], [27]).

These well-known solutions are 1-D, and for image processing and volume rendering they are extended to multiple dimensions through a separable extension (often called tensor-product approach) or through a spherical extension (e.g., McClellan transformation [22]). The problem of these extensions is that they do not deal well with the multi-dimensional nature of the sampling lattice, in particular for non-Cartesian lattices [19]. The separable extension clearly is only satisfactory for Cartesian lattices, for which the lattice's natural Nyquist region (see Section II) would coincide with the kernel's low-pass region in the frequency domain. The spherical extension, however, has difficulties imposing zero-crossing in the frequency domain at the dual-lattice points, which is crucial to guarantee approximation order and polynomial reproduction [30].

An interesting example in 2-D are “hex-splines”, which form a B-spline family for hexagonally sampled data [32]. The first-order hex-spline is defined as the indicator function of the Voronoi cell (and is thus non-separable). Higher-order hex-splines are defined in terms of successive convolutions of the first-order function. Analytical formulae are available both in the frequency and the spatial domain.

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Box splines offer a mathematically elegant framework for constructing a class of multi-dimensional elements with flexible shape and support that can be non-separable in a natural way. The general topic of box splines is rather intricate and a general survey of results on the topic has been gathered in [9]. In this paper we design a four-directional box spline that is geometrically aligned with the BCC lattice and allows for piecewise polynomial approximation of data sampled on the BCC lattice.

The most commonly used method for evaluating box splines at arbitrary points is through de Boor's recurrence relation [9]. Unfortunately, the recursive evaluation of box splines is computationally inefficient and prone to numerical instabilities [10]. Kobbelt addresses the instability issues by delaying the evaluation of the discontinuous step function until the latest stages of recursion [17]. Even though the numerical inaccuracies of the recursive algorithm can be minimized, to make box splines practical in the field of volume graphics (e.g., volume rendering), the computational complexity of their evaluation needs to be significantly reduced. While the use of box splines in surface subdivision in graphics demands evaluations of a box spline on a fixed mesh, in the volume rendering domain one needs to evaluate a box spline at arbitrary points. For traditional B-splines, the explicit piecewise polynomial representation is commonly used for fast evaluation; therefore, we introduce a new piecewise polynomial representation for the proposed box splines in Section V.

In [8], Dæhlen proposes an algorithm to evaluate a four-directional box spline on a fixed mesh shifted to an arbitrary position. Somewhat similar to our evaluation method, he relies on the relation of box splines to *cone splines* (truncated power functions). In Dæhlen's method, evaluation of truncated power functions is still based on a recurrence relation and is based on the connection with simplex splines. In our case, however, we derive the explicit polynomial representation of the truncated power function in Section V. This representation provides us with the exact evaluation of box splines free of numerical inaccuracies since we avoid any recurrence relations. Furthermore, similar to the piecewise polynomial evaluation methods of B-splines, our method exploits the symmetries in the support of the box spline to further reduce computational cost (see Section V-E.4).

In volume graphics, optimality of BCC sampling has been explored by Theußl et al. [20]. They applied the spherical extension of reconstruction filters, which resulted in rather blurry and unsatisfying results. Different ad-hoc approaches were studied for reconstruction and derivative reconstruction on the BCC lattice, with mixed results [19]. Also, isosurface extraction on the optimal sampling lattice has been studied with inconclusive results [4]. We also exploited the BCC lattice in multi-resolution analysis [14]. Recently, Csébfalvi [7] demonstrated a reconstruction using a Gaussian kernel and the principle of generalized interpolation [31]. While this method provides an isotropic solution, it does not guarantee approximation order. It is also a numerical scheme without any closed form solution for the interpolation kernel. Moreover, our explicit piecewise polynomial representation makes the box spline solution several times more computationally efficient than [7] (see Section VI).

### Scope and Organization of the Paper

In this paper, we provide accurate and efficient reconstruction methods for the BCC lattice. Such reconstructions have been sought for in the volume graphics community [4], [19], [20]

to better exploit the theoretical advantages of the BCC lattice. Several contributions are proposed:

- We establish a four-directional box spline that is geometrically tailored to the BCC lattice in Section IV<sup>1</sup>. The first-order box spline is a 3-D piecewise linear function. Higher-order versions are obtained by successive convolutions. This way, we can choose the required smoothness and approximation order.
- We explicitly characterize the polynomial patches defining these box splines, which is detailed in Section V. Our characterization method is general (for any order) and leads to polynomial expressions that can be implemented to evaluate the box spline at any point. Specifically, we derive the explicit expressions for the  $C^0$  and  $C^2$  members of our family of BCC-box-splines, since they are the most relevant for the practitioner in rendering applications.
- We demonstrate that our box splines (for  $C^0$  and  $C^2$  continuity) on the BCC lattice are computationally *twice* as efficient as tensor-product B-splines on the Cartesian lattice (for comparable smoothness and the same sampling density); see Table II. Based on these results, in Section VI, we conclude that BCC lattice sampling can be more attractive not only on a theoretical level, but also in a practical setting.

## II. GEOMETRY OF THE BCC LATTICE

A *lattice* is a periodic pattern made up by an infinite array of points in which each point has a neighborhood identical to those of all the other points [2]. In other words, every lattice point has the same Voronoi cell and we can refer to the Voronoi cell of the lattice without ambiguity.

Periodic sampling of a function in the spatial domain gives rise to a periodic replication of its spectrum in the frequency domain. The lattice that describes the centers of the replicas in the frequency domain is called the *dual*, *reciprocal*, or *polar* lattice. Reconstruction amounts to eliminating the replicas of the spectrum in the frequency domain and preserving the primary spectrum. Therefore, the ideal kernel for reconstruction in the space of “band-limited” functions is a function that is the inverse Fourier transform of the characteristic function of the Voronoi cell of the dual lattice.

In 3-D, the BCC lattice points are located on the corners of a cube with an additional sample in the center of this cube. Therefore, the BCC lattice can be considered as two interleaving Cartesian lattices where the vertices of the secondary Cartesian lattice are moved to the center of the primary Cartesian cells. An alternative way of describing the BCC lattice as a sub-lattice of the Cartesian is to start with a Cartesian lattice (i.e.  $\mathbb{Z}^3$ ) and retain only those points whose coordinates have identical parity. For an integer point in  $\mathbb{Z}^3$  to belong to the BCC lattice, all three of  $x$ ,  $y$  and  $z$  coordinates need to be odd or all three need to be even. Therefore, the BCC lattice is a subgroup whose quotient group is of order *four*<sup>2</sup>. Therefore, the BCC lattice is a sub-lattice of  $\mathbb{Z}^3$  whose density is  $1/4$  in  $\mathbb{Z}^3$ ; in other words the volume of the Voronoi cell of each lattice point is 4. This BCC lattice can be generated by integer linear combinations of the columns of the

<sup>1</sup>A preliminary version of these box splines was presented in our conference paper [13].

<sup>2</sup>Looking at the  $(x, y, z)$  coordinates mod 2, there are 8 different combinations and we are interested in only two:  $(0, 0, 0)$  and  $(1, 1, 1)$ .

sampling matrix:

$$BCC = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}. \quad (1)$$

Sampling a function with lattice points generated by the BCC sampling matrix, has the effect of replicating the spectrum of that function periodically on the dual lattice which is an FCC type lattice generated by [11]:

$$FCC = 2\pi BCC^{-T} = -\pi \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \quad (2)$$

The FCC lattice is often referred to as the  $D_3$  lattice [6]. In fact  $D_3$  belongs to a general family of lattices,  $D_n$ , sometimes called checkerboard lattices. The checkerboard property implies that the sum of the coordinates of the lattice sites is always even (multiple of  $\pi$  with the scaling in Equation 2). We will use this property to demonstrate the zero crossings of the frequency response of the reconstruction filters at the FCC lattice sites.

The simplest interpolation kernel on any lattice is the indicator function of the Voronoi cell of the lattice. The corresponding interpolation scheme is the generalization of the so-called nearest neighbor interpolation. More sophisticated reconstruction kernels involve information from the neighboring points of a given lattice point. With this in mind, we focus in the next section on the geometry and the polyhedra associated with the BCC lattice.

#### A. Polyhedra Associated with the BCC Lattice

Certain polyhedra arise naturally in the process of constructing interpolation filters for a lattice. The Voronoi cell of the lattice is one such example. The Voronoi cell of the Cartesian lattice is a cube and the Voronoi cell of the BCC lattice is a truncated octahedron as illustrated in Figure 1(a).

We are also interested in the cell formed by the immediate neighbors of a lattice point. The first (nearest) neighbors of a lattice point are defined via the Delaunay tetrahedralization of the lattice; a point  $q$  is a *first neighbor* of  $p$  if their respective Voronoi cells share a (non-degenerate) face. The *first neighbors cell* is the polyhedron whose vertices are the first neighbors. Again, this cell is the same for all points on the lattice.

For example, by this definition there are six first neighbors of a point in a Cartesian lattice; the first neighbors cell for the Cartesian lattice is the octahedron. For the BCC lattice there are fourteen first neighbors for each lattice point. The first neighbor cell is a rhombic dodecahedron as illustrated in Figure 1(b).

### III. BOX SPLINES REVIEW

Here, we begin by briefly introducing box splines and state their properties that will be useful further on.

A box spline is characterized by a set of direction vectors that indicate its construction by successive convolution of line segments along these vectors. The linear combination of shifts of a box spline generates a spline whose smoothness and ability to approximate continuous functions also depend on these direction vectors. Notationally, the direction vectors are usually gathered in a matrix; i.e., a box spline in  $\mathbb{R}^s$  is specified by  $n \geq s$  vectors in  $\mathbb{R}^s$  that are columns of its matrix  $\Xi = [\xi_1, \xi_2, \dots, \xi_n]$ . The support of the box spline is all points  $x \in \mathbb{R}^s$  such that  $x = \Xi t$  where  $t \in \mathbb{R}^n$  and  $0 \leq t_k \leq 1$  for  $1 \leq k \leq n$ . In other words, the

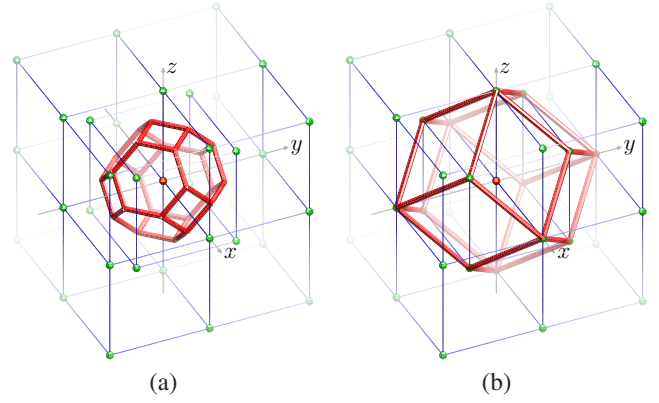


Fig. 1. The Voronoi cell of the BCC lattice is the truncated octahedron. (a) The first neighbors cell is the rhombic dodecahedron (b).

support of the box spline is contained in the convex combination of these direction vectors.

The simplest box spline is constructed by  $n = s$  vectors and is the (area-normalized) characteristic function of its support:

$$M_{\Xi}(x) = \begin{cases} \frac{1}{|\det \Xi|} & \text{where } x = \Xi t \text{ and } t \in [0, 1]^n \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Clearly, the box spline from Equation 3 is discontinuous at the boundary of its support. Its 1-D version is the *boxcarr* function that is simply the indicator function for the interval  $[0, 1]$ .

For the general case  $n > s$ , the box splines are defined recursively:

$$M_{[\Xi, \xi]}(x) = \int_0^1 M_{\Xi}(x - t\xi) dt. \quad (4)$$

This inductive definition implies that starting from the base case as in Equation 3 the indicator function is smeared *along* the additional direction vector. Hence, the convolution of two box splines is yet another box spline:

$$M_{\Xi_1} * M_{\Xi_2}(x) = M_{[\Xi_1, \Xi_2]}(x). \quad (5)$$

A box spline is a piecewise polynomial of degree at most  $n - s$ . Moreover, let  $\rho$  be the minimal number of vectors such that, if they were removed from  $\Xi$ , the remaining vectors would *not* span  $\mathbb{R}^s$ . Then  $M_{\Xi} \in C^{\rho-2}$ , where  $C^n$  is the space of  $n$ -times continuously differentiable functions. The Fourier transform of a box spline is:

$$\hat{M}_{\Xi}(\omega) = \prod_{\xi \in \Xi} \frac{1 - \exp(-i\xi^T \omega)}{i\xi^T \omega} \quad (6)$$

where  $i = \sqrt{-1}$  as usual. In 2-D, the simplest box spline is specified by:

$$\Xi_0 = [\xi_1 \quad \xi_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is the indicator function of the unit square  $[0, 1]^2$ .

Adding a direction vector of  $\xi_3 = [1 \ 1]^T$  to  $\Xi_0$  smears the unit square across its diagonal. This is illustrated in Figure 2. As the basic box spline is a constant function on the unit square, the result of smearing it along the diagonal produces a linear box spline that is represented by  $[\Xi_0, \xi_3]$ . The support of this box spline is illustrated in Figure 2(b). This box spline is a bivariate piecewise polynomial of degree one. This box spline generates a  $C^0$  spline function space, as  $\rho = 2$ .

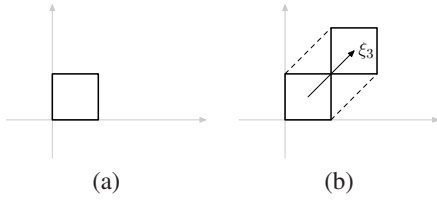


Fig. 2. Construction of the linear element from the simplest box spline.

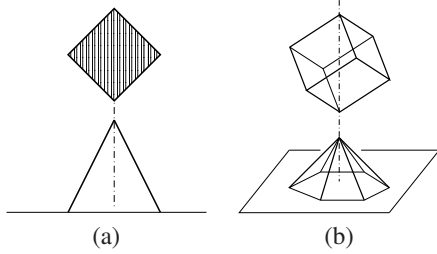


Fig. 3. (a) 1-D linear box spline (triangle function). (b) 2-D hexagonal linear box spline.

#### IV. FOUR-DIRECTION BOX SPLINE ON THE BCC LATTICE

The construction of box splines dedicated to the BCC lattice is guided by the fact that the rhombic dodecahedron (the first neighbors cell of the BCC lattice, see Figure 1(b)) is the 3-D shadow of a 4-D hypercube (tesseract) along its antipodal axis. This construction is a generalization of the 2-D linear box spline with hexagonal support, which can be obtained by projecting a 3-D cube along its antipodal axis; see Figure 3(b).

##### A. Geometric Construction

Integrating a constant tesseract along its antipodal axis yields a function that has a rhombic dodecahedron support (see Figure 1(b)), has its maximum value at the center and has a *linear* falloff towards the fourteen first-neighbor vertices. Since it arises from the projection of a higher dimensional box, this function serves as the linear box spline interpolation kernel on the BCC lattice.

Let  $\mathcal{B}$  denote the boxcar function. The characteristic function of the unit tesseract is given by a tensor-product of four  $\mathcal{B}$  functions on each axis. By projecting the unit tesseract, one obtains a rhombic dodecahedron whose geometric scale is only half of the first-neighbours cell of the BCC lattice described by Equation 1. In this BCC lattice, with integer lattice coordinates, the first-neighbors cell is scaled such that the blue edges of Figure 1(b) are of length two. Therefore, we scale the geometry of the unit tesseract by two and normalize by its hypervolume:

$$\mathcal{T}(x, y, z, w) := \frac{1}{16} \mathcal{B}(x/2) \mathcal{B}(y/2) \mathcal{B}(z/2) \mathcal{B}(w/2). \quad (7)$$

Let  $\mathbf{v} = (2, 2, 2, 2) := [2, 2, 2, 2]^T$  denote a vector along the antipodal axis. In order to project along this axis, it is convenient to rotate it so that it aligns with the  $w$  axis:

$$\mathbf{R} = \frac{1}{2} [\rho_1 \rho_2 \rho_3 \rho_4] = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \quad (8)$$

This rotation matrix transforms  $\mathbf{v}$  to  $(0, 0, 0, 4)$ . Also by examining Equation 8, one can see that each vertex of the rotated tesseract, when projected along the  $w$  axis, will coincide with

the origin or one of the vertices of the rhombic dodecahedron:  $(\pm 1, \pm 1, \pm 1)$ ,  $(\pm 2, 0, 0)$ ,  $(0, \pm 2, 0)$ ,  $(0, 0, \pm 2)$  or  $(0, 0, 0)$ . Let  $\mathbf{x} = (x, y, z, w)$ ; now the linear box spline is given by

$$L_{RD}(x, y, z) = \frac{1}{16} \int \mathcal{T}(\mathbf{R}^{-1} \mathbf{x}) dw.$$

Substituting in Equation 7 we get

$$L_{RD}(x, y, z) = \frac{1}{16} \int \prod_{k=1}^4 \mathcal{B}\left(\frac{1}{4} \rho_k^T \mathbf{x}\right) dw. \quad (9)$$

Note that value at the origin is  $L_{RD}(0, 0, 0) = 1/4$  (see [9, II.8]). This is due to the fact that the box splines are normalized to  $\int L_{RD}(\mathbf{x}) d\mathbf{x} = 1$ , whereas the sampling density of the BCC lattice ( $\frac{1}{4}$ ) demands a kernel whose integral is 4. Therefore, in order to preserve the energy in the discrete/continuous model we employ the box splines scaled by 4 on the BCC lattice. This scaling ensures that the value of the linear box spline at the center is 1 and zero at all other lattice sites. Hence the linear box spline constitutes a linear interpolator on the BCC lattice.

##### B. Fourier Transform

If the direction matrix of a box spline is known, the distributional definition of box splines easily leads to their frequency domain representation. Here we present a geometric derivation of the Fourier transform of our box spline.

From the projection-based construction of the rhombic dodecahedron discussed earlier, we can derive the Fourier transform of the linear box spline function described by Equation 9. From Equation 7, it is evident that the Fourier transform of the characteristic function of the tesseract is given by the tensor-product:

$$\hat{\mathcal{T}}(\omega_x, \omega_y, \omega_z, \omega_w) = \frac{1 - \exp(-i2\omega_x)}{i2\omega_x} \frac{1 - \exp(-i2\omega_y)}{i2\omega_y} \frac{1 - \exp(-i2\omega_z)}{i2\omega_z} \frac{1 - \exp(-i2\omega_w)}{i2\omega_w}$$

since  $\frac{1}{2} \mathcal{B}(x/2) \longleftrightarrow \frac{1 - \exp(-i2\omega)}{i2\omega}$ . We use  $\longleftrightarrow$  to indicate a Fourier transform pair.

By the Fourier slice-projection theorem, projecting a function along a certain direction in the spatial domain amounts to slicing its Fourier transform perpendicular to the direction of projection. This slice runs through the origin. Again we make use of the rotation (8) to align the projection axis with the  $w$  axis. Thus in the frequency domain we take the slice  $\omega_w = 0$ .

It is convenient to introduce the  $3 \times 4$  matrix

$$\Xi = [\xi_1 \xi_2 \xi_3 \xi_4] = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \quad (10)$$

given by the first three rows of the rotation matrix  $\mathbf{R}$  of Equation 8. The reason for omitting the last row is that we are taking a slice  $\omega_w = 0$  orthogonal to the fourth axis at the origin. The Fourier transform of the linear box spline can now be written as

$$\hat{L}_{RD}(\omega_x, \omega_y, \omega_z) = \prod_{k=1}^3 \frac{1 - \exp(-i\xi_k^T \boldsymbol{\omega})}{i\xi_k^T \boldsymbol{\omega}}$$

where  $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ . The space domain function  $L_{RD}$  corresponds to the box spline  $M_\Xi$ ; we will use this box spline



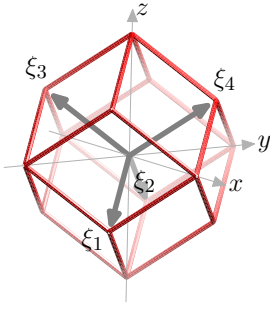


Fig. 4. The support of the box spline represented by  $\Xi$  is a rhombic dodecahedron formed by the four direction vectors in  $\Xi$ .

symbol from now on. The Fourier transform of this box spline is then:

$$\hat{M}_{\Xi}(\omega) = \prod_{k=1}^4 \frac{1 - \exp(-i\xi_k^T \omega)}{i\xi_k^T \omega}. \quad (11)$$

Since any three of the vectors in  $\Xi$  span  $\mathbb{R}^3$ , at least two vectors need to be removed from  $\Xi$  so the remaining vectors do not span, therefore  $\rho = 2$ . Hence this box spline is guaranteed to produce a  $C^0$  reconstruction. We can verify the vanishing moments (zero crossings) of the frequency response at the aliasing-frequencies on the FCC lattice points. We first note that  $\sum_{k=1}^4 \xi_k = \mathbf{0}$ ; therefore, the center of the box spline  $M_{\Xi}$  is at the origin [9] and the Fourier transform can be written as:

$$\hat{M}_{\Xi}(\omega) = \prod_{k=1}^4 \text{sinc}(\xi_k^T \omega).$$

Recall that  $\text{sinc}(t) = \frac{\sin(t/2)}{t/2}$ . This re-formulation provides a more convenient form to verify zero crossings. Due to the checkerboard property, for every FCC lattice point the sum of its coordinates is always even. Since the FCC lattice dual to the BCC lattice of our discussion is scaled by  $\pi$  (according to Equation 1), for  $\omega$  on the FCC lattice,  $\xi_4^T \omega = (\omega_x + \omega_y + \omega_z) = 2\pi k$  for some  $k \in \mathbb{Z}$ ; therefore,  $\text{sinc}(\xi_4^T \omega) = 0$  on all of the aliasing-frequencies. Since  $\xi_4^T \omega = -\xi_1^T \omega - \xi_2^T \omega - \xi_3^T \omega$ , at least one of the  $\xi_m^T \omega$  for  $m = 1, 2, 3$  needs to also be an even multiple of  $\pi$  since the sum of three odd multiples of  $\pi$  can not be an even multiple. For such  $k$  we have  $\text{sinc}(\xi_k^T \omega) = 0$ ; therefore, there is a zero of order at least two at each aliasing-frequency, yielding a  $C^0$  kernel whose approximation power is two on the BCC lattice [30]. This smoothness and approximation power parallels that of the trilinear B-spline interpolation on the Cartesian lattice.

### C. Higher-Order Box Splines

The number of vanishing moments can be doubled by convolving the linear box spline with itself. Hence the resulting reconstruction kernel will have twice the approximation power on the BCC lattice due to the Strang-Fix result [30]. As noted before, the resulting box spline can then be represented by  $\Xi^2 := [\Xi, \Xi]$  where every direction vector is duplicated.

An equivalent method of deriving this function would be to convolve the constant function on the tesseract with itself and project the resulting distribution along a diagonal axis (this commutation of convolution and projection is easy to understand in terms of the corresponding operators in the frequency domain – see Section IV-B). Convoluting the constant function on the tesseract with itself results in another function supported on a

tesseract which is the tensor-product of four 1-D triangle (linear B-spline) functions. Let  $\Lambda$  denote the triangle function. Then the convolution yields

$$\mathcal{T}_c(x, y, z, w) = \frac{1}{16} \Lambda\left(\frac{1}{2}x\right) \Lambda\left(\frac{1}{2}y\right) \Lambda\left(\frac{1}{2}z\right) \Lambda\left(\frac{1}{2}w\right). \quad (12)$$

Following the same 4-D rotation as in the previous section, we obtain a space domain representation of the new box spline:

$$C_{RD}(x, y, z) = \frac{1}{16} \int \prod_{k=1}^4 \Lambda\left(\frac{1}{4} \rho_k^T x\right) dw. \quad (13)$$

Similar to the linear case, we use the matrix of  $\Xi^2$  to represent this properly scaled box spline. Since convolution in the space domain amounts to a multiplication in the frequency domain, we use Equation 11 to derive the Fourier transform of the new box spline:

$$\hat{M}_{\Xi^2}(\omega) = \hat{M}_{\Xi}^2(\omega) = \prod_{k=1}^4 \left( \frac{1 - \exp(-i\xi_k^T \omega)}{i\xi_k^T \omega} \right)^2. \quad (14)$$

We can see that the number of vanishing moments of this box spline are doubled when compared to the linear kernel. This implies that this box spline has fourth order approximation power on the BCC lattice [30]. The eight directions of this box spline  $\Xi^2$  are duplicates of the original four directions. Consequently, the minimum number of directions that one needs to remove from  $\Xi^2$  so that the remaining vectors do not span  $\mathbb{R}^3$  is  $\rho = 4$ ; hence, weighted shifts of this box spline are guaranteed to produce  $C^2$  continuous reconstructions with fourth order approximation. This smoothness and approximation power parallels that of the tricubic B-spline reconstruction on the Cartesian lattice; for this reason we have referred to  $M_{\Xi^2}$  as the ‘cubic’ box spline in [13]. However, since there are eight directions, this trivariate box spline is composed of quintic polynomials. Therefore according to de Boor’s notations [9], we will call this box spline a quintic box spline.

As we noted earlier  $M_{\Xi}$  is of second order approximation power on the BCC lattice. The  $n^{\text{th}}$  convolution of the linear kernel with itself, denoted by  $M_{\Xi^n}$ , will have approximation power of  $2n$  on the BCC lattice. These box splines would produce  $C^{2n-2}$  reconstructions.

### D. Support

The support of  $M_{\Xi}$  is a rhombic dodecahedron as shown in Figure 4. The support of  $M_{\Xi^n}$  is the Minkowski sum of  $n$  copies of rhombic dodecahedron. Since a rhombic dodecahedron is a convex and symmetric polyhedron (with respect to its center), its Minkowski addition with itself will have the same shape, scaled by two. In general the support of  $M_{\Xi^n}$  would be a rhombic dodecahedron scaled by  $n$  [33].

The volume of the support of the box spline  $M_{\Xi}$  as depicted in Figure 4 is 16. Therefore, for a point  $x$  in general position, 16 points from  $\mathbb{Z}^3$  intersect the support of  $M_{\Xi}(x)$  [9, II.15]. Since only 1/4 of these points belong to the BCC lattice, only 4 BCC points fall inside the support of  $M_{\Xi}(x)$ . Similarly, the support of  $M_{\Xi^2}$  is a rhombic dodecahedron whose direction vectors are scaled by two. Therefore, its volume is 128 which implies that only 32 BCC points fall inside the support of  $M_{\Xi^2}(x)$ .

TABLE I

RECONSTRUCTION PROPERTIES OF PROPOSED BOX SPLINES ON THE BCC LATTICE AND TENSOR-PRODUCT B-SPLINES ON THE CARTESIAN LATTICE.

	Cartesian B-splines		BCC Box Splines	
	Trilinear	Tricubic	Linear	Quintic
Degree of continuity	$C^0$	$C^2$	$C^0$	$C^2$
Order of approximation	2	4	2	4
Degree of polynomial pieces	3	9	1	5
# Data points in the support	8	64	4	32

This fact implies that a  $C^0$  reconstruction with a second order approximation power on BCC only needs 4 data points<sup>3</sup>, while for this smoothness and accuracy on the Cartesian lattice, trilinear interpolation requires a neighborhood of  $2 \times 2 \times 2 = 8$  data points. Similarly, a  $C^2$  reconstruction with a fourth order approximation on BCC only needs 32 data points, while for this smoothness and accuracy on the Cartesian lattice, tricubic B-spline requires a neighborhood of  $4 \times 4 \times 4 = 64$  data points (see Table I for a summary). Hence, as we will see in Section VI, the computational cost of BCC reconstruction is significantly lower than a similar reconstruction on the Cartesian lattice with an equivalent sampling density.

## V. EXPLICIT PIECEWISE POLYNOMIAL REPRESENTATION

The previous section defined the four-direction box spline on the BCC lattice and showed some of its main properties derived from its Fourier transform. However, a literal implementation of Equation 9 and Equation 13, as we implemented in [13], turned out to be extremely inefficient (especially in the case of the quintic box spline). Hence, although theoretically exciting, these splines were not useful in a practical setting. To make them practical for computer graphics and visualization applications, we derive a piecewise polynomial representation that allows an extremely fast evaluation as desired for these applications.

### A. Preliminaries and Outline of Derivation

In the following discussion, the symbol  $\nabla_{\xi}$  denotes a (directional) *finite-difference operator* and is defined by:  $\nabla_{\xi} f(x) = f(x) - f(x - \xi)$ . For a matrix of directions,  $\Xi$ , the difference operator is defined as successive applications of difference operators along each direction in  $\Xi$ :  $\nabla_{\Xi} = \prod_{\xi \in \Xi} \nabla_{\xi}$ . The corresponding differential operator is denoted by  $D_{\Xi}$ . Green's function of a differential operator is a function  $g$  that satisfies  $Dg = \delta$  where  $\delta$  denotes Dirac's delta (generalized) function. The Fourier transform of  $\delta$  in the distributional sense is the constant function 1.

Box splines, similar to B-splines, are piecewise polynomial functions with bounded support. In this section we will see that the box spline  $M_{\Xi}$  can be derived by applying the finite-difference operator,  $\nabla_{\Xi}$ , to a single function  $G_{\Xi}$  which is the Green's function for the differential operator  $D_{\Xi}$  corresponding to  $\nabla_{\Xi}$ .

The essential idea in our derivation is to closely analyze the numerator and denominator of the Fourier transform of box splines (as in Equation 11). The numerator corresponds to the

box spline's difference operator in the space domain, which is defined as

$$\nabla_{\Xi} \longleftrightarrow \prod_{\xi \in \Xi} 1 - \exp(-i\xi^T \omega). \quad (15)$$

In Section V-B, we will derive the weights, and their respective positions in 3-D, as a discrete series for the finite difference operator of our box splines.

Using distribution theory, we can identify the remaining part of Equation 11 as the Fourier transform of  $G_{\Xi}$  in space domain, since

$$D_{\Xi} \longleftrightarrow \prod_{\xi \in \Xi} i\xi^T \omega, \text{ and } G_{\Xi} \longleftrightarrow \prod_{\xi \in \Xi} \frac{1}{i\xi^T \omega}.$$

If this differential operator  $D_{\Xi}$  is applied to its Green's function  $G_{\Xi}$ , Dirac's delta is obtained. However, if we apply the corresponding finite difference operator to the Green's function, the box splines are obtained:

$$M_{\Xi}(x) = \nabla_{\Xi} G_{\Xi}(x).$$

We will show in Section V-C that the function  $G_{\Xi}$  is constructed by superpositions and linear transformations of a tensor-product of (two-sided) signed monomials:  $x^k \operatorname{sgn}(x)$ . In Section V-D we will see that we can also derive box splines by applying the difference operators on their *truncated powers*,  $T_{\Xi}$ . Truncated powers are very similar to  $G_{\Xi}$  but instead of the two-sided signed monomials, they are constructed from one-sided monomials:

$$(x)_+^k = \begin{cases} x^k & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \quad (16)$$

$(x)_-^k$  is also defined as  $(x)_-^k = x^k - (x)_+^k$ . Since one-sided monomials are supported on half-spaces, they are more convenient than Green's functions in derivations.

Four-direction box splines of general higher degrees are obtained by  $n$  convolutions of the linear box spline, which amounts to:

$$M_{\Xi^n}(x) = \nabla_{\Xi^n} T_{\Xi^n}(x). \quad (17)$$

These box splines are represented in the frequency domain by:

$$\hat{M}_{\Xi^n}(\omega) = \hat{M}_{\Xi}^n(\omega) = \frac{1}{i^{4n}} \prod_{k=1}^4 \frac{(1 - \exp(-i\xi_k^T \omega))^n}{(\xi_k^T \omega)^n}.$$

For notational convenience we introduce the scalar variables:

$$\begin{aligned} z_k &:= \exp(-i\xi_k^T \omega) \\ w_k &:= \xi_k^T \omega. \end{aligned} \quad (18)$$

This notation allows to write the Fourier transforms of higher degree box splines more compactly as:

$$\hat{M}_{\Xi^n}(\omega) = \prod_{k=1}^4 \frac{(1 - z_k)^n}{w_k^n}.$$

Furthermore, we note that due to the structure of  $\sum_{k=1}^4 \xi_k = 0$  in  $\Xi$ ,  $\sum_{k=1}^4 w_k = 0$  and  $\prod_{k=1}^4 z_k = 1$ .

<sup>3</sup>The four points in the linear interpolant construct a barycentric interpolation on the tetrahedron they form.

### B. Difference Operator

The finite difference operator can be represented as a filter. Its coefficients weight the Green's function that is shifted to the various lattice points, as in Equation 17. The Z-domain representation of the difference operator allows for an easy polynomial representation of this discrete series:

$$\nabla_{\Xi}^n(z) = \prod_{k=1}^4 (1 - z_k)^n. \quad (19)$$

Expanding this equation for the linear box spline  $n = 1$ , and using the fact that  $\prod_{k=1}^4 z_k = 1$ , we get:

$$\begin{aligned} \nabla_{\Xi}(z) = & 2 - (z_1 + z_2 + z_3 + z_4) \\ & + (z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4) \\ & - (z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4). \end{aligned} \quad (20)$$

For a more compact notation, we adopt a slightly different multinomial notation where the power operation on 4-tuples  $Z = (z_1, z_2, z_3, z_4)$  by  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  is defined as:

$$Z^{\alpha} = \frac{1}{p(\alpha)} \sum_{(\beta_1, \dots, \beta_4) \in \text{perm}(\alpha_1, \dots, \alpha_4)} z_1^{\beta_1} z_2^{\beta_2} z_3^{\beta_3} z_4^{\beta_4} \quad (21)$$

where  $\text{perm}(\alpha)$  is the set of all permutations of  $\alpha$  and  $p(\alpha)$  counts the number of permutations of repeated values in  $\alpha$ . This is to avoid counting duplicate terms of the polynomial. For instance if the value of  $\alpha_1$  is repeated in  $\alpha$   $r_1$  times and the value of  $\alpha_2$  is repeated  $r_2$  times, then  $p(\alpha) = r_1! r_2!$ . In this notation, the difference operator weights are represented as:

$$\nabla_{\Xi}(z) = 2 - Z^{(1,0,0,0)} + Z^{(1,1,0,0)} - Z^{(1,1,1,0)}.$$

Note that since  $z_1^{-1} = z_2 z_3 z_4$ , both  $Z^{(1,0,0,0)}$  and  $Z^{(1,1,1,0)}$  denote the same set of monomials which contain exactly one lattice vector. Geometrically, one can visualize the weights being 2 at the origin, -1 on all BCC lattice points reachable by exactly one lattice vector (positive or negative) and +1 on lattice points that can be reached by exactly two lattice vectors. This is illustrated in Figure 5(a). Similarly the difference operator weights for the quintic box spline  $n = 2$  can be derived from Equation 19 in the compact form as:

$$\begin{aligned} \nabla_{\Xi}^2(z) = & 18 - 10Z^{(1,0,0,0)} + 8Z^{(1,1,0,0)} + 4Z^{(2,1,1,0)} \\ & - 2Z^{(2,1,0,0)} + Z^{(2,0,0,0)} + Z^{(2,2,0,0)}. \end{aligned} \quad (22)$$

These weights are on BCC lattice points on a rhombic dodecahedron with twice larger neighborhood than that of the linear box spline case and are displayed in Figure 5(b).

### C. Green's Function

We now describe a procedure to derive the space domain representation of the Green's functions of our box splines. We make use of the  $w_k$  variables introduced in Equation 18:

$$\hat{G}_{\Xi}(\omega) = \prod_{k=1}^4 \frac{1}{\xi_k^T \omega} = \prod_{k=1}^4 \frac{1}{w_k}.$$

The objective is to re-write  $\hat{G}_{\Xi}$  into a number of terms, each of which contains only three of the four  $w_k$  variables. Such a three-variable expression can then be written as a linear transformation of a trivariate function whose inverse Fourier transform can be obtained by a tensor-product. The general idea is to exploit the

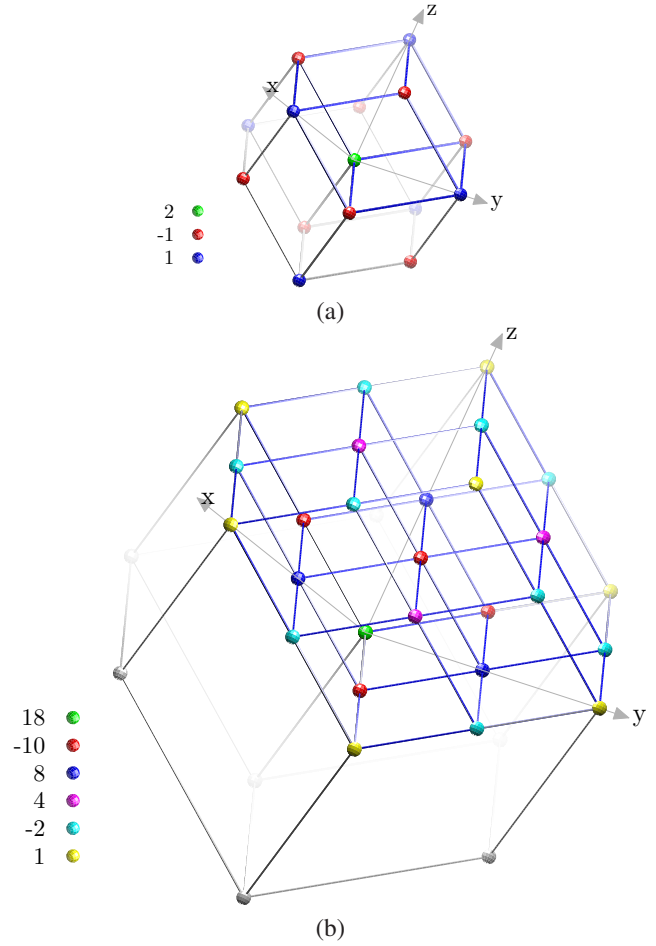


Fig. 5. (a) Weights for the difference operator of the linear box spline. (b) Weights for the difference operator of the quintic box spline. For simplicity of illustration only one of the parallelepipeds constituting the rhombic dodecahedron has been drawn with all of its internal vertices.

relation  $w_1 + w_2 + w_3 = -w_4$  to reduce the number of variables in the denominator and obtain a sum of terms with one less variable. This helps to eliminate any fourth variable with the help of the proper numerator and consequently we introduce new terms in the expression while increasing the power of  $w_4$ . This procedure is the frequency domain reasoning of the spatial domain recursive structure of the box splines. We can always apply this procedure since whenever the number of directions  $n$  is greater than the dimension of the space  $s$ , the additional directions of the box spline can be written as the linear combination of the  $s$  linearly independent vectors. The  $\hat{G}_{\Xi}$  of the linear box spline can be re-written as:

$$\begin{aligned} \hat{G}_{\Xi}(\omega) &= \frac{1}{w_1 w_2 w_3 w_4} = -\frac{\frac{w_1 + w_2 + w_3}{w_4}}{w_1 w_2 w_3 w_4} \\ &= \frac{-1}{w_4^2 w_1 w_2 w_3} \times (w_1 + w_2 + w_3). \end{aligned}$$

While the general Green's function of the  $n$ th box spline is  $\hat{G}_{\Xi^n} = (\hat{G}_{\Xi})^n$ , for the linear box spline we have  $n = 1$ :

$$\hat{G}_{\Xi}(\omega) = \frac{-1}{w_4^2} \left( \frac{1}{w_1 w_2} + \frac{1}{w_1 w_3} + \frac{1}{w_2 w_3} \right). \quad (23)$$

Similarly the quintic box spline's Green's function is obtained by  $n = 2$ :

$$\begin{aligned} \hat{G}_{\Xi^2}(\omega) &= \hat{G}_{\Xi}^2(\omega) = \\ &= \frac{1}{w_4^4} \left( \frac{1}{w_1^2 w_2^2} + \frac{1}{w_1^2 w_3^2} + \frac{1}{w_2^2 w_3^2} \right) + \\ &= \frac{-2}{w_4^5} \left( \frac{1}{w_1^2 w_3} + \frac{1}{w_1^2 w_2} + \frac{1}{w_2^2 w_3} + \frac{1}{w_2^2 w_1} + \frac{1}{w_3^2 w_1} + \frac{1}{w_3^2 w_2} \right) + \\ &= \frac{6}{w_4^6} \left( \frac{1}{w_2 w_3} + \frac{1}{w_1 w_3} + \frac{1}{w_1 w_2} \right). \end{aligned} \quad (24)$$

Now we can move back to using the frequency variables  $\omega = (\omega_x, \omega_y, \omega_z)$ . We first define these building-block functions:

$$\begin{aligned} \hat{\rho}_1(\omega_x, \omega_y, \omega_z) &= \frac{-1}{\omega_x \omega_y \omega_z^2} \\ \hat{\rho}_2(\omega_x, \omega_y, \omega_z) &= \frac{1}{\omega_x^2 \omega_y^2 \omega_z^4} - 2 \left( \frac{1}{\omega_x^2 \omega_y} + \frac{1}{\omega_x \omega_y^2} \right) \frac{1}{\omega_z^5} + \frac{6}{\omega_x \omega_y \omega_z^6}. \end{aligned} \quad (25)$$

These functions are useful since the  $\hat{G}_{\Xi}$  of our box splines are essentially linear transformations (e.g.  $(w_1, w_2, w_4) = \Xi_{\{1,2,4\}}^T \omega$ ) and superpositions of these building-block functions:

$$\begin{aligned} \hat{G}_{\Xi}(\omega) &= \hat{\rho}_1(\Xi_{\{1,2,4\}}^T \omega) + \hat{\rho}_1(\Xi_{\{1,3,4\}}^T \omega) + \hat{\rho}_1(\Xi_{\{2,3,4\}}^T \omega) \\ \hat{G}_{\Xi^2}(\omega) &= \hat{\rho}_2(\Xi_{\{1,2,4\}}^T \omega) + \hat{\rho}_2(\Xi_{\{1,3,4\}}^T \omega) + \hat{\rho}_2(\Xi_{\{2,3,4\}}^T \omega). \end{aligned}$$

Here the subscript  $\{i, j, k\}$  indicates the matrix formed by the  $i, j$  and  $k$ th columns of  $\Xi$ .

We now derive the inverse Fourier transform of these building-block functions. First, we recognize that the Fourier inverse of  $1/(i\omega)^k$  is the two-sided monomial [1]:

$$\frac{(x)_{\text{sgn}}^k}{k!} := \frac{1}{2} \frac{x^k \text{sgn}(x)}{k!} \longleftrightarrow \frac{1}{(i\omega)^{k+1}}. \quad (26)$$

We can derive the space domain representation of our building-block functions as a tensor-product inverse Fourier transform of equations in 25:

$$\begin{aligned} \rho_1(x, y, z) &= - (x)_{\text{sgn}}^0 (y)_{\text{sgn}}^0 (z)_{\text{sgn}} \\ \rho_2(x, y, z) &= \frac{1}{3!} (x)_{\text{sgn}} (y)_{\text{sgn}} (z)_{\text{sgn}}^3 + \\ &= \frac{-2}{4!} \left[ (x)_{\text{sgn}} (y)_{\text{sgn}}^0 (z)_{\text{sgn}}^0 + (x)_{\text{sgn}}^0 (y)_{\text{sgn}} (z)_{\text{sgn}}^0 \right] (z)_{\text{sgn}}^4 + \\ &= \frac{6}{5!} (x)_{\text{sgn}}^0 (y)_{\text{sgn}}^0 (z)_{\text{sgn}}^5. \end{aligned} \quad (27)$$

If  $Q$  is an invertible matrix, we know that  $f(Qx) \longleftrightarrow \hat{f}((Q^{-1})^T \omega) / |\det Q|$ . Therefore, we can write the space domain representation of the Green's function of these box splines as:

$$\begin{aligned} G_{\Xi}(x) &= \frac{1}{4} \left( \rho_1(\Xi_{\{1,2,4\}}^{-1} x) + \rho_1(\Xi_{\{1,3,4\}}^{-1} x) + \rho_1(\Xi_{\{2,3,4\}}^{-1} x) \right) \\ G_{\Xi^2}(x) &= \frac{1}{4} \left( \rho_2(\Xi_{\{1,2,4\}}^{-1} x) + \rho_2(\Xi_{\{1,3,4\}}^{-1} x) + \rho_2(\Xi_{\{2,3,4\}}^{-1} x) \right) \end{aligned} \quad (28)$$

where  $x = (x, y, z)$  and  $|\det \Xi_{\{1,2,4\}}^{-1}| = |\det \Xi_{\{1,3,4\}}^{-1}| = |\det \Xi_{\{2,3,4\}}^{-1}| = 1/4$ .

#### D. Truncated Power

Recall that the Green's functions were constructed from the signed monomials  $x^k \text{sgn}(x)$  in equations 27. The differential

operator  $D_{\Xi}$ , when applied on these signed monomials, transformed in Equation 28, results in a  $\delta$  function. Consequently, this differential operator annihilates all *polynomials* of degree  $\leq k$  encountered in the signed polynomials in the Green's function. Similarly the finite-difference operator annihilates all of these polynomials [9, I.32].

Since box splines are obtained by applying the difference operator on the Green's function, we can add or subtract any polynomials up to degree  $k$  found in the Green's function. Therefore, the box spline can also be obtained by applying the difference operator to a Green's function that is obtained from  $\frac{1}{2}(x^k \text{sgn}(x) - x^k) = -(x)_+^k$  or  $\frac{1}{2}(x^k \text{sgn}(x) + x^k) = (x)_+^k$ . The contributions from adding or subtracting  $x^k$  is eliminated since the difference operator annihilates the polynomials made of  $x^k$ . The advantage of working with these one-sided monomials is that they are supported on half-spaces; whereas the support of Green's functions in Equation 28 is the entire space.

Therefore, we redefine the building-block functions, by using  $(x)_+^k, (y)_+^k$  and  $-(z)_-^k$ :

$$\begin{aligned} \tau_1(x, y, z) &= (x)_+^0 (y)_+^0 (z)_- \\ \tau_2(x, y, z) &= \frac{-1}{3!} (x)_+ (y)_+ (z)_-^3 + \\ &= \frac{2}{4!} \left[ (x)_+ (y)_+^0 + (x)_+^0 (y)_+ \right] (z)_-^4 + \\ &= \frac{-6}{5!} (x)_+^0 (y)_+^0 (z)_-^5. \end{aligned} \quad (29)$$

Note that the support of these building block functions is on points  $x \in \mathbb{R}^3$  such that  $x, y > 0$  and  $z < 0$ . We further derive the truncated power functions to be:

$$\begin{aligned} T_{\Xi}(x) &= \frac{1}{4} \left( \tau_1(\Xi_{\{1,2,4\}}^{-1} x) + \tau_1(\Xi_{\{1,3,4\}}^{-1} x) + \tau_1(\Xi_{\{2,3,4\}}^{-1} x) \right) \\ T_{\Xi^2}(x) &= \frac{1}{4} \left( \tau_2(\Xi_{\{1,2,4\}}^{-1} x) + \tau_2(\Xi_{\{1,3,4\}}^{-1} x) + \tau_2(\Xi_{\{2,3,4\}}^{-1} x) \right) \end{aligned} \quad (30)$$

The crucial point here is that the values of truncated power functions at any point  $x$ , are affected only by one of the three terms on the right hand sides of the above equations. To see this fact recall that  $\tau_1(x, y, z)$  and  $\tau_2(x, y, z)$  are non-zero only when  $x, y > 0$  and  $z < 0$ . The support of each building-block function is transformed in Equation 30 to cones formed by columns of  $\Xi$  (e.g. the support of  $\tau_1(\Xi_{\{1,2,4\}}^{-1} x)$  is all points  $x = \Xi_{\{1,2,4\}}(t_1, t_2, t_3)$  for  $t_1, t_2 > 0$  and  $t_3 < 0$ ). The support of the building-block functions is transformed as:

$$\begin{aligned} \Xi_{\{1,2,4\}}^{-1} x &= \frac{1}{2} (x - z, y - z, x + y) \\ \Xi_{\{1,3,4\}}^{-1} x &= \frac{1}{2} (x - y, z - y, x + z) \\ \Xi_{\{2,3,4\}}^{-1} x &= \frac{1}{2} (y - x, z - x, y + z). \end{aligned} \quad (31)$$

Therefore, the support of  $\tau_1(\Xi_{\{1,2,4\}}^{-1} x)$  is the cone that is the intersection of the half-spaces determined by  $x - z > 0, y - z > 0$  and  $x + y < 0$  (see Figure 6). Similarly, the support of  $\tau_1(\Xi_{\{1,3,4\}}^{-1} x)$  is the cone that is the intersection of the half-spaces determined by  $x - y > 0, z - y > 0$  and  $x + z < 0$ . Since  $y - z < 0$  and  $z - y < 0$  are disjoint, the support of  $\tau_1(\Xi_{\{1,2,4\}}^{-1} x)$  and  $\tau_1(\Xi_{\{1,3,4\}}^{-1} x)$  are disjoint. Therefore, the supports of each building-block function transformed in Equations 30 are non-overlapping.

The support of the truncated power functions of the box splines is the union of each cone formed by the matrices of the equations



in 30. The support of each transformed building-block function along with the support of the truncated power of the box splines is illustrated in Figure 6. The red arrows indicate the half-spaces in the support. Therefore, one can verify that the support of the truncated power functions is the *union* of half-spaces determined by  $x + y < 0$ ,  $x + z < 0$  and  $y + z < 0$ .

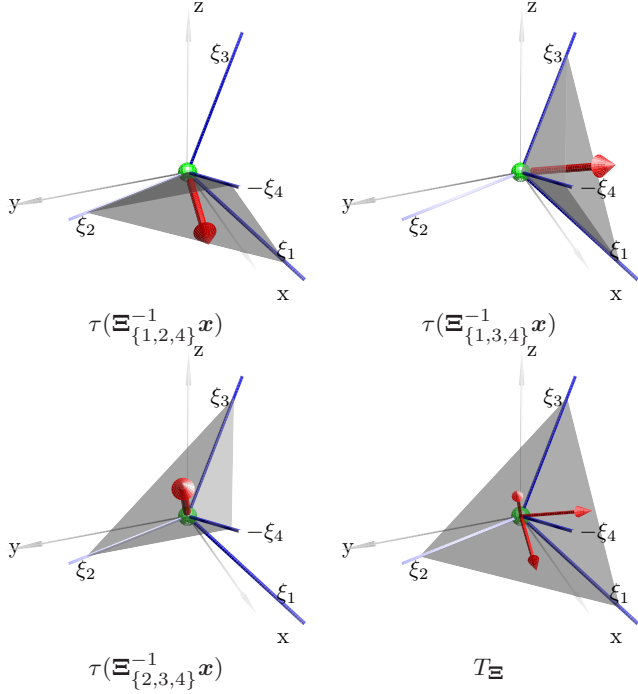


Fig. 6. The support of the truncated power function is the cone formed by the three directions in  $\Xi_{\{1,2,3\}}$ . This volume is a *disjoint union* of support of three  $\tau$  functions each transformed by  $\Xi_{\{1,2,4\}}^{-1}$ ,  $\Xi_{\{1,3,4\}}^{-1}$  and  $\Xi_{\{2,3,4\}}^{-1}$ .

Since the supports of the transformed  $\tau$  functions are non-overlapping, at any point  $x$  only one of the three transformed  $\tau$ 's contribute to the value of the truncated power. For  $x$  to be within the support of one of the transformed  $\tau$  functions its first and second components of the transformed vector need to be positive while the third component needs to be negative. Under this assumption the last component of each vector of the right hand side of Equation 31 is the sum of the two largest values out of  $x$ ,  $y$  and  $z$ . And the other two components in each set is the difference of the largest and the middle value from the minimum of the three. We also notice that these basic building-block functions are symmetric with respect to the first and second components of the position vector  $x$  of their argument (see Equation 27). For example when  $\tau$  is transformed by  $\Xi_{\{1,2,4\}}^{-1}$ , its support is determined by the region specified by  $x > z$ ,  $y > z$  and  $x + y < 0$ . Using these observations we can write the truncated power in terms of only one basic building-block function:

$$T_{\Xi}(x, y, z) = \frac{1}{4} \tau_1\left(\frac{1}{2}(\tilde{x} - \tilde{z}), \frac{1}{2}(\tilde{y} - \tilde{z}), \frac{1}{2}(\tilde{x} + \tilde{y})\right) \quad (32)$$

$$T_{\Xi^2}(x, y, z) = \frac{1}{4} \tau_2\left(\frac{1}{2}(\tilde{x} - \tilde{z}), \frac{1}{2}(\tilde{y} - \tilde{z}), \frac{1}{2}(\tilde{x} + \tilde{y})\right) \quad (33)$$

where

$$\tilde{x} = \max(x, y, z), \quad \tilde{y} = \text{mid}(x, y, z) \quad \text{and} \quad \tilde{z} = \min(x, y, z). \quad (34)$$

### E. Efficient Evaluation

Having obtained the explicit form of the truncated power, we shall apply the difference operator derived in Section V-B to  $T_{\Xi^n}$  in order to obtain  $M_{\Xi^n}$ . This operation indicated as in Equation 17 can be implemented as a convolution of the finite difference operator sequence with the truncated power. In this section we exploit the symmetries in the support of these box splines and find a *region* for efficient evaluation of this convolution.

1) *Region of evaluation: 1-D example:* To better understand the procedure we use to derive the polynomial pieces of the box spline, we first illustrate this procedure in 1-D for a linear B-spline ( $\Xi = [1 \ -1]$ ). In this case, the Green's function is  $-\frac{1}{2}x \text{sgn}(x)$  and the truncated power is  $T(x) = (x)_-$ . The difference operator is represented in the Z-domain by  $\nabla(z) = -z^{-1} + 2 - z$ . The linear B-spline is obtained by the following convolution:

$$\Lambda(x) = -T(x+1) + 2T(x) - T(x-1).$$

The process of this convolution is illustrated in Figure 7. In this figure, (a) shows the truncated power  $(x)_-$ , (b) shows the difference operator weights, (c) the result of the convolution illustrated by overlaying the truncated powers at their respective difference operator *sites* and (d) shows the resulting B-spline. In Figure 7(d) the red band indicates the regions of the  $x$  axis that are affected by the convolution site at 1, the green band indicates the region that is affected by the convolution site at 0 and the blue band indicates the region that is affected by the convolution site at  $-1$ . The symmetry of the support of the linear B-spline suggests an efficient evaluation in the interval  $[0, 1]$  where only one convolution site contributes to the values of the B-spline in this region.

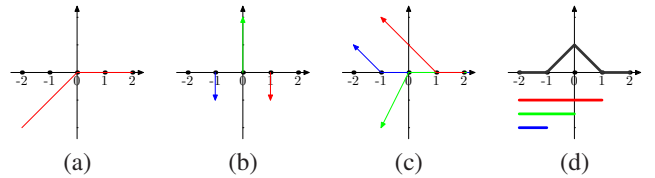


Fig. 7. Convolution of the truncated power  $(x)_-$  with the difference operator, a 1-D example. (a) the truncated power  $(x)_-$ . (b) the discrete difference operator weights. (c) the convolution by overlaying the truncated power functions. (d) the resulting convolution yields the linear B-spline.

Therefore, for an efficient evaluation of the linear B-spline we would map any point in  $[-1, 1]$ , the support of the B-spline, to the interval  $[0, 1]$  using the symmetry of its support. Once this mapping is performed, the B-spline can be computed by evaluation of the truncated power shifted only to the sites that affect this region. In the case of the linear B-spline there is only one site that affects this region which is  $T(x-1)$ .

2) *Region of evaluation: trivariate case:* Since our trivariate box splines are obtained through a projection along the antipodal axis of a tesseract they exhibit the symmetries present in their polyhedral support which is a rhombic dodecahedron. We exploit the symmetries present within the rhombic dodecahedron to achieve an efficient evaluation method for the linear and quintic box splines.

First, we observe that a rhombic dodecahedron can be decomposed into four non-overlapping parallelepipeds in two different ways. For a rhombic dodecahedron formed by the vectors in

$\Xi$  as in Figure 4, one can construct four parallelepipeds each formed by three of the four vectors from  $\Xi$ . Alternatively, one can choose the negative directions from  $-\Xi$  to decompose the rhombic dodecahedron into four parallelepipeds. Therefore, we can confine the evaluation region to one of these parallelepipeds and the evaluation at the other points can be inferred by symmetry.

The support of  $T_{\Xi}$  (or  $T_{\Xi^2}$ ) is the positive cone of  $\Xi_{\{1,2,3\}}$  as in Figure 6. A minimal number of convolution sites contribute to the value of the box spline in the parallelepiped which is cornered at the origin and formed by  $-\xi_1, -\xi_2$  and  $-\xi_3$ . This parallelepiped contains the positive octant of  $\mathbb{R}^3$  and is illustrated with blue edges in Figure 8. Similarly for the quintic box spline, a minimal number of convolution sites contribute to the parallelepiped formed by  $-2\xi_1, -2\xi_2$  and  $-2\xi_3$  which is illustrated in Figure 9.

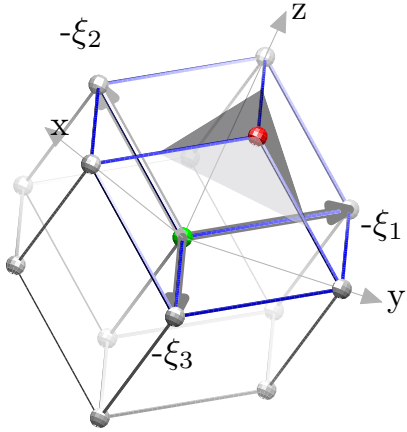


Fig. 8. When computing the operation of the difference operator on the truncated power, only one term which shifts the truncated power to  $(1, 1, 1)$  affects the parallelepiped of interest formed by  $-\Xi_{\{1,2,3\}}$ . The support of the truncated powers shifted to the other sites, do not intersect this parallelepiped.

3) *Linear Box Spline*: The operation of the difference operator on the truncated power is a sum of truncated power functions shifted and weighted according to the difference operator sites as in Figure 5(a). As the support of the truncated power is limited to the cone of the direction vectors in  $\Xi_{\{1,2,3\}}$ , only one of the terms of the convolution contributes to the value of  $M_{\Xi}$  in the parallelepiped that we deal with; this term is the one obtained from shifting the  $T_{\Xi}$  to  $(1, 1, 1)$  and multiplying by  $-1$ , the difference operator weight at this point (Figure 5(a)). None of the other difference operator sites affect this region of interest, as illustrated in Figure 8.

Therefore, a point in the parallelepiped of focus is characterized by  $(x, y, z) = -\Xi_{\{1,2,3\}}(t_1, t_2, t_3)$  where  $0 \leq t_1, t_2, t_3 < 1$ . Using the min/mid/max variables introduced in Equation 34 we have:

$$\begin{aligned} M_{\Xi}(x, y, z) &= (-1)T_{\Xi}(x-1, y-1, z-1) \\ &= -\frac{1}{4}\tau_1 \begin{pmatrix} 1/2(\tilde{x}-1-(\tilde{z}-1)), \\ 1/2(\tilde{y}-1-(\tilde{z}-1)), \\ 1/2(\tilde{x}-1+\tilde{y}-1) \end{pmatrix} \\ &= -\frac{1}{8}(\tilde{x}+\tilde{y}-2) \\ &= \frac{1}{4}\left(1 - \frac{1}{2}(\max(x, y, z) + \text{mid}(x, y, z))\right), \end{aligned} \quad (35)$$

which agrees with the geometric simplification we derived in [13] normalized according to the sampling density of the BCC lattice

described by Equation 1.

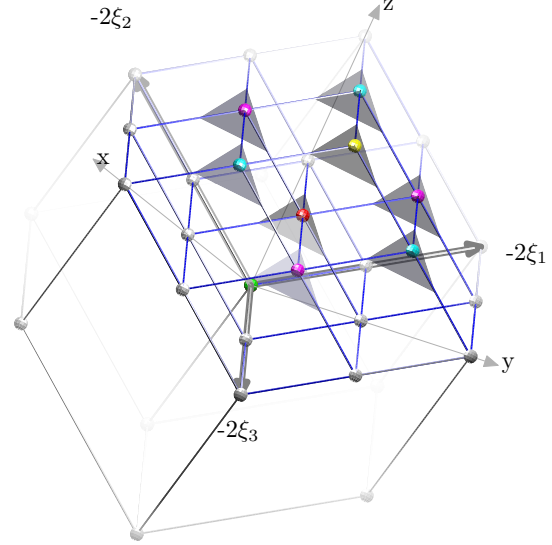


Fig. 9. When computing the convolution of the difference operator and the truncated power, only eight terms of the convolution affect the parallelepiped of interest formed by  $-2\Xi_{\{1,2,3\}}$ . The support of the truncated power shifted to each site is the cone indicated by the gray region. The support of the truncated power shifted to the other convolution sites, does not intersect this parallelepiped.

4) *Quintic Box Spline*: The same procedure as for the linear box spline can be used. The difference operator for the quintic box spline is shown in Figure 5(b). The support of the truncated power is limited to the cone of the direction vectors in  $\Xi_{\{1,2,3\}}$  such that only eight of the terms of the convolution contribute to the value of  $M_{\Xi^2}$  in the parallelepiped that we focus on; see Figure 9. As we saw in Section IV-D the size of the support of the quintic box spline is doubled from that of the linear box spline; therefore, the parallelepiped of focus is now eight times the size of the corresponding one in the linear box spline case; therefore for a point in the parallelepiped of focus characterized by  $(x, y, z) = -\Xi_{\{1,2,3\}}(t_1, t_2, t_3)$  where  $0 \leq t_1, t_2, t_3 < 2$ , we have:

$$\begin{aligned} M_{\Xi^2}(x, y, z) &= T_{\Xi^2}(x-2, y-2, z-2) \\ &\quad - 10T_{\Xi^2}(x-1, y-1, z-1) - 2T_{\Xi^2}(x-3, y-1, z-1) \\ &\quad - 2T_{\Xi^2}(x-1, y-3, z-1) - 2T_{\Xi^2}(x-1, y-1, z-3) \\ &\quad + 4T_{\Xi^2}(x-2, y-2, z) + 4T_{\Xi^2}(x-2, y, z-2) \\ &\quad + 4T_{\Xi^2}(x, y-2, z-2) \end{aligned}$$

where  $T_{\Xi^2}$  is defined as in Equation 33. The shifts in the above equation are shifts to the difference operator sites which are the colored nodes in Figure 9.

Using the symmetries of the rhombic dodecahedron we can confine the evaluation region to a tetrahedron that has a vertex at the origin and its apex (which is on the plane  $x+y=4$ ) is a quarter of one face of the original parallelepiped since these faces are rhomboids and have four-fold symmetry. The rhomboid face of the parallelepiped of our focus lies in the plane specified by  $x+y=4$  in Figure 10. Out of four possible choices, we pick this tetrahedron so that it contains the positive octant completely. This region is specified by its four bounding planes:  $x+y < 4$ ,  $x > y$ ,  $y > z$  and  $z > 0$ . This tetrahedron is illustrated by the dark tetrahedron in Figure 10. It is partitioned into four regions formed by the intersections with four of the eight sub-parallelepipeds

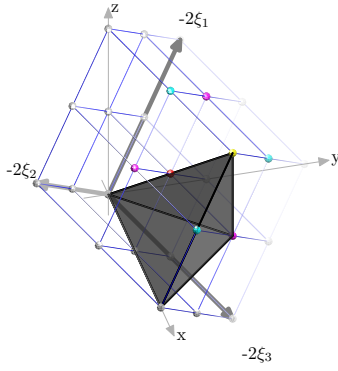


Fig. 10. The region specified by  $x + y < 4$  and  $x > y > z > 0$  is illustrated with the dark tetrahedron. This tetrahedron is formed by connecting the origin to the face which is the triangle that is one quarter of the rhombic face of the original rhombic dodecahedron.

that constitute the original parallelepiped of focus. These sub-parallelepipeds are highlighted in green in Figure 11. These four regions are identified by:

$$\begin{aligned} R_1 : x + y < 2; & \quad R_2 : x + y > 2, x + z < 2; \\ R_3 : x + z > 2, y + z < 2; & \quad R_4 : y + z > 2. \end{aligned}$$

These regions are determined by posing the restriction of being in the dark tetrahedron of focus which is specified by  $x > y > z > 0$  and each of the four sub-parallelepipeds that intersect this tetrahedron. These sub-parallelepipeds are specified by  $-\Xi_{\{1,2,3\}}t$  where the sub-parallelepiped of region  $R_1$  is specified by  $0 \leq t_1, t_2, t_3 < 1$ ; the sub-parallelepiped of region  $R_2$  is specified by  $0 \leq t_1, t_2 < 1$  and  $1 \leq t_3 < 2$ ; the sub-parallelepiped of region  $R_3$  is specified by  $0 \leq t_1 < 1$  and  $1 \leq t_2, t_3 < 2$  and the sub-parallelepiped of region  $R_4$  is specified by  $1 \leq t_1, t_2, t_3 < 2$ .

In each of these regions, illustrated in Figure 11, the box spline will be represented as a separate polynomial.

The region  $R_1$  is affected by all of the eight difference operator sites in the parallelepiped of focus. Region  $R_2$  is affected only by four sites at  $(3, 1, 1)$ ,  $(2, 2, 2)$ ,  $(1, 3, 1)$  and  $(2, 2, 0)$ . Region  $R_3$  is affected by two sites at  $(3, 1, 1)$  and  $(2, 2, 2)$ . Finally, region  $R_4$  is affected only by  $(2, 2, 2)$ . Therefore, we simplify the polynomials in each region separately. Using constants  $\alpha := 1/3840$ ,  $\beta := 1/1920$  and  $\gamma := 1/960$  we have:

Region  $R_1$ ,  $M_{\Xi^2}(x, y, z) =$

$$\begin{aligned} & \alpha(x + y - 4)^3(-3xy - 5z^2 + 2x + 2y + 20z + x^2 + y^2 - 24) \\ & + \beta(x + z - 2)^3(x^2 - 9x - 3xz + 10y - 5y^2 + 14 + 11z + z^2) \\ & + \beta(y + z - 2)^3(46 - 30x - z - y + 3zy + 5x^2 - y^2 - z^2) \\ & - \gamma(x + y - 2)^3(x^2 + x - 3xy - 5z^2 + y^2 + y - 6). \end{aligned} \quad (36)$$

Region  $R_2$ ,  $M_{\Xi^2}(x, y, z) =$

$$\begin{aligned} & \alpha(x + y - 4)^3(-3xy - 5z^2 + 2x + 2y + 20z + x^2 + y^2 - 24) \\ & - \beta(x + z - 2)^3(-z^2 - 11z + 3xz - 14 + 5y^2 + 9x - 10y - x^2) \\ & - \beta(y + z - 2)^3(-46 + z + 30x + y - 3zy - 5x^2 + y^2 + z^2). \end{aligned} \quad (37)$$

Region  $R_3$ : This region as illustrated in Figure 11 is not a simple tetrahedron with homogeneous regions with respect to the site located at  $(3, 1, 1)$ . When the truncated power is centered at this

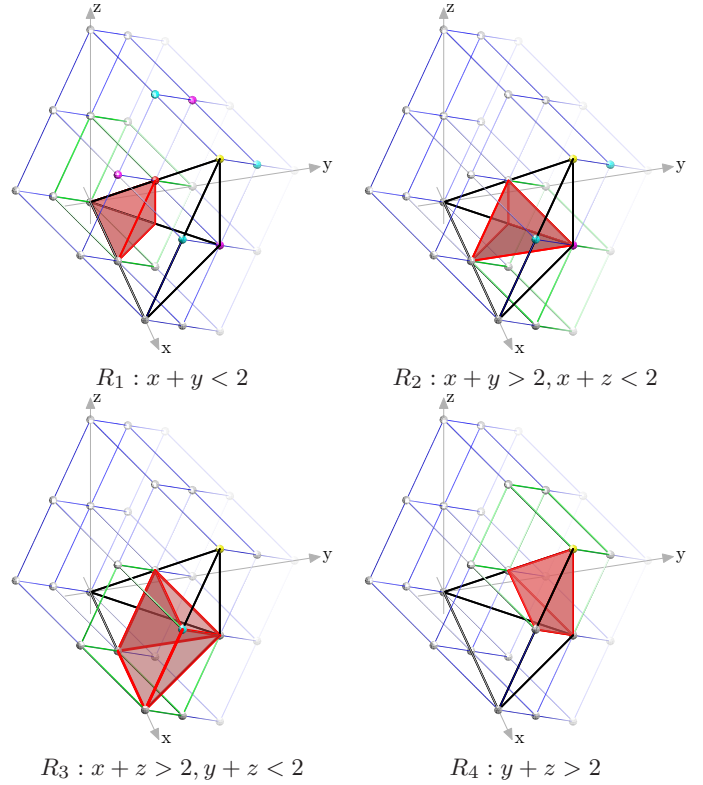


Fig. 11. Different regions of the evaluation domain are the intersection of sub-parallelepipeds (highlighted in green) and the tetrahedron in focus (black edges). The box spline is a homogeneous polynomial in each region.

site, two of the three components constituting the truncated power (see Figure 6) intersect this region. Therefore, there are two sub-cases for the contribution of the truncated power centered at  $(3, 1, 1)$ . But the contribution from  $(2, 2, 2)$  remains homogeneous as only one of the three components of the truncated power (see Figure 6) contributes to this region:

Region  $R_{3A}\{x - z > 2\}$ ,  $M_{\Xi^2}(x, y, z) =$

$$\alpha(x + y - 4)^3(-x^2 + 8x + 3xy - y^2 + 5z^2 - 16 - 12y). \quad (38)$$

Region  $R_{3B}\{x - z < 2\}$ ,  $M_{\Xi^2}(x, y, z) =$

$$\begin{aligned} & \alpha(x + y - 4)^3(-3xy - 5z^2 + 2x + 2y + 20z + x^2 + y^2 - 24) \\ & - \beta(y + z - 2)^3(30x + z - 46 - 3yz + y - 5x^2 + y^2 + z^2). \end{aligned} \quad (39)$$

Region  $R_4$ ,  $M_{\Xi^2}(x, y, z) =$

$$\alpha(x + y - 4)^3(-3xy - 5z^2 + 2x + 2y + 20z + x^2 + y^2 - 24). \quad (40)$$

Therefore a fast evaluation of the box spline can be obtained for any  $(x, y, z)$  by first transforming it to the tetrahedron of focus (the dark tetrahedron in Figure 10) by taking  $(x, y, z) = (|x|, |y|, |z|)$  and then sorting  $(x, y, z) = (\max(x, y, z), \text{mid}(x, y, z), \min(x, y, z))$ . Once these transforms are performed, we can test for the appropriate of the above five regions and the polynomial form can be evaluated. We also note that due to the explicit piecewise polynomial form above, and the relatively small support of these polynomial regions, our evaluation is numerically stable.

As a summary we include the pseudo-code for the evaluation of the quintic box spline referring to the regions above.



```

function fourdirection_boxspline(x,y,z)
% Transform the point to the tetrahedron of focus in Figure 10
x = abs(x); y = abs(y); z = abs(z);
sort x,y,z in decreasing order
% For the linear box spline, the apex of the focus-tetrahedron is
% on the plane (x+y)=2: the boundary
if ((x+y)>2) return 0;
return (2-(x+y))/8; % Linear case; see Equation 35

% For the quintic box spline the plane (x+y)=4 is the boundary:
if ((x+y)>4) return 0;
if ((x+y)<2) % Region R1
    return MΞR1(x,y,z) as in Equation 36;
elseif ((x+z)<2) % Region R2
    return MΞR2(x,y,z) as in Equation 37;
elseif ((y+z)<2) % Region R3
    if ((x-z)>2) % Region R3A
        return MΞR3A(x,y,z) as in Equation 38;
    else % Region R3B
        return MΞR3B(x,y,z) as in Equation 39;
else % Region R4
    return MΞR4(x,y,z) as in Equation 40;
end

```

## VI. RESULTS

In this section we describe our experimental results and compare the BCC sampling scheme to the traditional Cartesian sampling on volumetric datasets. In order to examine the reconstruction schemes discussed in this paper, we have implemented a ray-caster to render images from the Cartesian and the BCC sampled volumetric datasets. We have chosen the synthetic dataset (Figure 12) first proposed in [21] as a benchmark for our comparisons. The function was sampled at the resolution of  $41 \times 41 \times 41$  on the Cartesian lattice and at an almost<sup>4</sup> equivalent sampling on the BCC lattice of  $32 \times 32 \times 64$ . The images in Figure 13 are rendered using the quintic box spline on the BCC sampled datasets and the tricubic B-spline on the Cartesian sampled datasets. The analytical function was rendered by evaluating the actual function proposed in [21]. The images in the second row in Figure 13 document the corresponding error images that are obtained of the angular error that occurred in estimating the normal by central differencing on the reconstructed function. Although direct reconstruction of the normal is possible by using the analytical gradient of the reconstruction kernel, we chose central differencing with a relatively small step on the reconstructed function to approximate the true gradient. Central differencing is commonly the method of choice in the visualization domain and there is no reason to believe that it performs any better or worse than taking the analytical derivative of the reconstruction kernel [26].

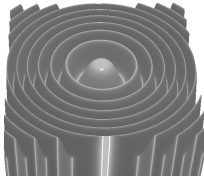


Fig. 12. The explicit function introduced by Marschner-Lobb.

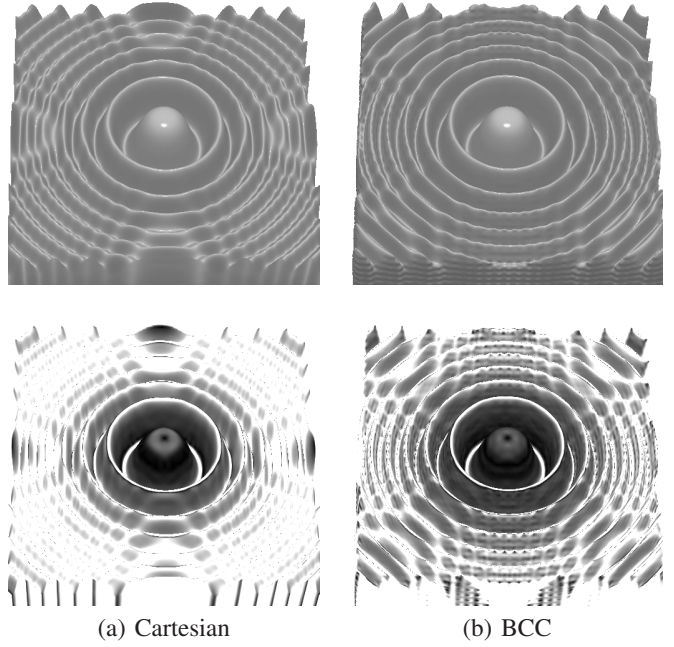


Fig. 13. The Marschner-Lobb dataset. (a) Sampled on the Cartesian lattice at the resolution of  $41 \times 41 \times 41$ . (b) Sampled on the BCC lattice at the resolution of  $32 \times 32 \times 64$ . The first row illustrates the volume rendering of the sampled data using the tricubic B-spline on the Cartesian and our quintic box spline on the BCC dataset. The second row illustrates the corresponding angular errors in estimating the gradient on the iso-surface from the reconstruction process. An angular error of .3 radians is mapped to white. The darker error image of the BCC data confirms smaller errors and more accurate reconstruction.

The gray value of 255 (white) denotes an angular error of .3 radians between the computed normal and the exact normal. The superiority of the BCC sampling is apparent by comparing the images in Figure 13(a) and Figure 13(b) as these are obtained from an equivalent sampling density over the volume. While the lobes are mainly preserved in the BCC case, they are more smoothed out in the case of Cartesian sampling. This is also confirmed by their corresponding error images in the second row of Figure 13.

Typical volumetric data are scanned on and reconstructed from the Cartesian lattice. A suitable (anti-aliasing) pre-filtering step is applied to limit the spectrum of the sampled data within the Nyquist region, which is the Voronoi cell of the reciprocal lattice. For the BCC lattice, this cell is clearly different from the one of the Cartesian lattice. Therefore, the ultimate test of the BCC reconstruction on real-life datasets can not be performed until there are true BCC sampling scanners available.

Nevertheless, we constructed comparable BCC and Cartesian datasets by merely subsampling a fairly densely sampled Cartesian dataset. Cartesian sampled data can be downsampled onto a BCC lattice by retaining Cartesian points whose  $x, y, z$  coordinates are all odd or all even. Such a BCC lattice has a quarter of the sampling density of the original dataset. For obtaining an almost equivalent subsampling ratio into a lower resolution Cartesian dataset, we choose a rational subsampling scheme where each dimension of the original Cartesian dataset is subsampled by  $63/100$  since  $(63/100)^3 = 0.250047 \approx 1/4$ . To achieve this subsampling, we first upsampled by zero-padding in the frequency domain by a factor of 63. Then a subsampling of the rate  $1/100$  yields the properly subsampled Cartesian volume.

<sup>4</sup>Since a finite sampling of a volume can not produce the exact same number of samples for the BCC and Cartesian sampling patterns, for our discrete resolutions, we chose the resolutions conservatively in favor of the Cartesian sampling. Therefore, the actual sampling density in the Cartesian sampled datasets is slightly higher than the BCC sampling density.



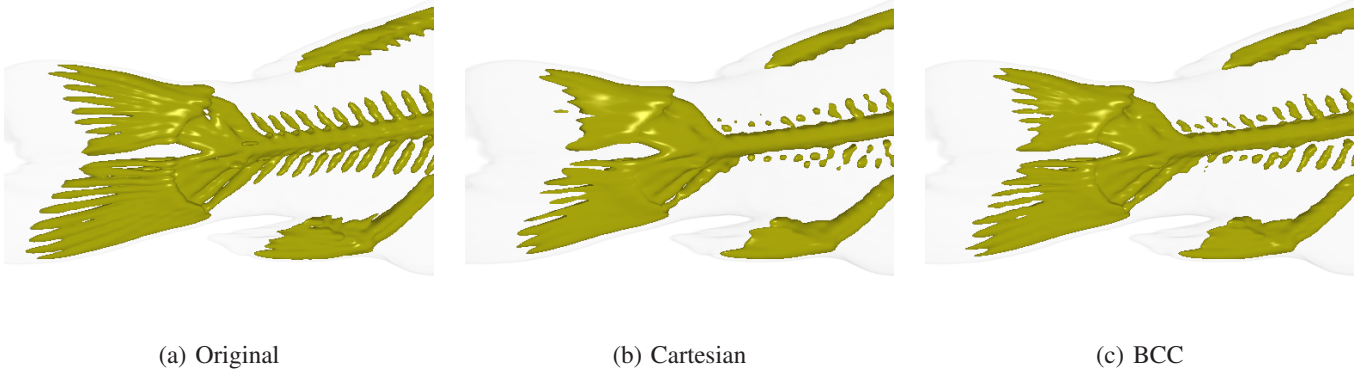


Fig. 15. The carp fish dataset. (a) The original Cartesian sampled dataset with 16,777K samples reconstructed with the tricubic B-spline. (b) Undersampled on the Cartesian lattice with 2,744K samples reconstructed with the tricubic B-spline. (c) Undersampled on the BCC lattice with 2,735K samples reconstructed with the quintic box spline.

As a first practical case, we chose the Boston Teapot dataset. The original dataset has a resolution of  $162 \times 162 \times 113$ . The subsampled Cartesian volume has a resolution of  $103 \times 103 \times 72$  and the subsampled BCC volume has a resolution of  $81 \times 81 \times 113$ . These volumes were rendered using the tricubic B-spline on the Cartesian lattice and the quintic box spline on the BCC lattice in Figure 14. These images demonstrate the superiority of the BCC sampling scheme since the Cartesian undersampled dataset developed cracks on the surface of the teapot lid, while the BCC undersampled dataset maintains the original content with higher fidelity. We also examined the carp fish dataset. The original dataset has a resolution of  $256 \times 256 \times 256$ . The subsampled Cartesian volume has a resolution of  $140 \times 140 \times 140$  and the subsampled BCC volume has a resolution of  $111 \times 111 \times 222$ . These volumes were also rendered with the tricubic B-spline and the quintic box spline respectively; see Figure 15. Again, these results show the superiority of the BCC sampling scheme since the Cartesian undersampled dataset misses the fish tail and most of the bones.

In [13] we have discussed the issues pertaining to linear order interpolation. While in the Cartesian volumes they demonstrate grid-aligned artifacts, in BCC they display girdering artifacts [4]. The result of linear box spline on the BCC and trilinear B-spline interpolation on the Cartesian lattice are demonstrated in Figure 16.

We have also approximated the mean square error existing in the volumes subsampled on BCC and Cartesian lattices. The error calculation was carried out by a random sampling of the error and summing over these random points to approximate the  $L_2$  error integral. These experiments also confirmed that the BCC subsampling is more accurate than the comparable Cartesian subsampling since the error of the Cartesian subsampled volume matched that of the BCC volume with only about 70% of the number of samples. Further, we have examined the visual quality of the rendered images and found empirical evidence that a BCC sampled volume with roughly about 70% of the number of samples of a Cartesian volume produces equivalent visual quality [23].

*Computational Cost:* The computational cost of the reconstruction is mainly due to computing the convolution of the data

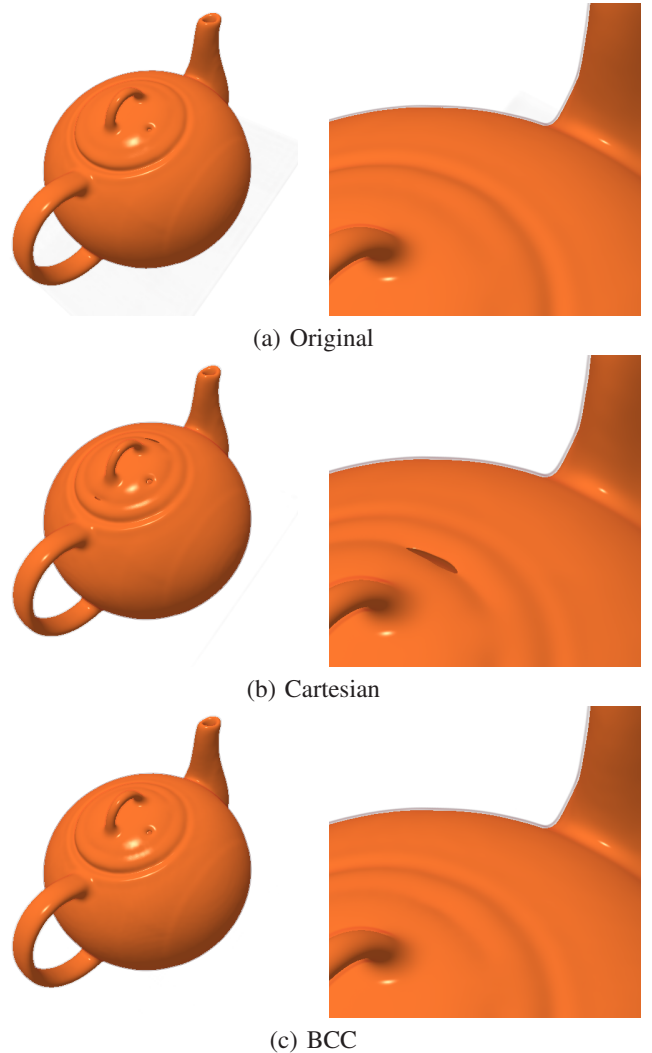


Fig. 14. The Boston teapot dataset. (a) The original Cartesian sampled dataset with 2,965K samples reconstructed with the tricubic B-spline. (b) Undersampled on the Cartesian lattice with 763K samples reconstructed with the tricubic B-spline. The surface shows an artifact. (c) Undersampled on the BCC lattice with 741K samples reconstructed with the quintic box spline.

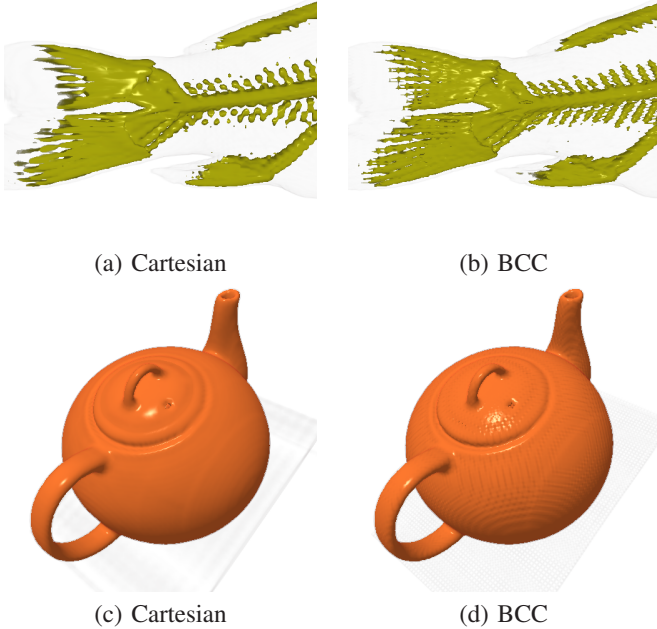


Fig. 16. Trilinear B-spline versus linear box spline reconstructions. (a) and (b) are the  $C^0$  reconstruction of volumes in Figure 15. (c) and (d) are the  $C^0$  reconstruction of volumes in Figure 14.

values and the continuous-domain box spline kernel. For trilinear and tricubic B-spline reconstructions on the Cartesian lattice, a neighborhood of  $2 \times 2 \times 2 = 8$  and  $4 \times 4 \times 4 = 64$  points fall inside the support of the kernels, respectively. Therefore, eight terms of the convolution in the case of the trilinear and sixty-four terms in the case of the tricubic B-spline need to be computed. Computing the convolution weights involves evaluating a *third* degree trivariate polynomial for trilinear while a *ninth* degree trivariate polynomial needs to be evaluated for the tricubic B-spline. However, due to the tensor-product structure of these kernels, the third degree polynomial, in the case of the trilinear interpolation, factors into a product of three first degree univariate polynomials. Similarly, the ninth degree trivariate polynomial of the cubic B-spline factors into a product of three third degree univariate polynomials.

For linear and quintic box spline reconstructions on the BCC lattice, a neighborhood of 4 and 32 points fall inside the support of the kernels, respectively. Therefore, only four terms of the convolution in the case of linear and thirty two terms in the case of the quintic box spline need to be computed. Computing the convolution weights involves evaluating a *first* degree trivariate polynomial for the linear box spline while a *fifth* degree trivariate polynomial needs to be evaluated for the quintic box spline. However, due to the structure of the quintic kernel, the fifth degree polynomial is factored into the product of a third degree polynomial and a second degree polynomial as Equation 27 can be factored in terms of the  $z$  variable. All of the polynomial pieces of the quintic box spline listed in Section V-E.4 are in the form of this building block function.

Our experiments also support these predictions as the Cartesian dataset in Figure 13(a) was rendered in 122.69 seconds while the BCC dataset in Figure 13(b) was rendered in 63.75 seconds. These images were computed at a resolution of  $500 \times 500$  on a dual

TABLE II  
RENDERING TIMES.  $C^0$  AND  $C^2$  INDICATE THE LINEAR AND QUINTIC BOX SPLINES ON THE BCC LATTICE AND THE TRILINEAR AND TRICUBIC B-SPLINES ON THE CARTESIAN LATTICE RESPECTIVELY.

	Marschner-Lobb		Carp		Teapot	
	$C^0$	$C^2$	$C^0$	$C^2$	$C^0$	$C^2$
BCC	11.99	63.75	33.85	184.33	29.05	150.45
Cartesian	21.49	122.69	60.08	363.62	50.27	294.55
Speedup	1.79	1.92	1.77	1.97	1.73	1.96

processor (Dual Core AMD Opteron 280) machine running Linux with a GCC compiler (4.0.2). A similar rendition using trilinear interpolation on the Cartesian image took 21.49 seconds while the linear box spline on the BCC took 11.99 seconds. Similar timing differences were observed on the real-life datasets; the timings for these reconstructions are summarized in Table II. We note that for  $C^0$  reconstructions, the speedups are less than a factor of two. Since linear interpolation is relatively light, a smaller portion of the rendering time is consumed by the reconstruction step; hence, twice a speedup in reconstruction plays a slightly less significant role in the rendering time.

## VII. CONCLUSION

In this paper, for the first time, we have demonstrated the practicality of box splines as well as BCC lattice sampling. This is a significant contribution in two areas. First, it takes the mathematically elegant construct of box splines and shows that they are practical and can be efficiently implemented. While the derivation in this paper is for a specific class of box splines, we believe that its principle can be extended to general box splines. Second, while BCC lattice sampling has been known to be theoretically superior over (standard) Cartesian sampling, it has not received the attention it deserves from practitioners – mainly due to the absence of computational tools. This paper makes a major step in bringing the theoretical advantages of the BCC lattice to a solid and efficient practical foundation and should pave the way to a mainstream adoption of alternative sampling structures. We plan to investigate higher-degree box splines for efficient prefiltering (interpolation and quasi-interpolation); see [5] for some recent advances on the hexagonal lattice.

## ACKNOWLEDGMENTS

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